The Problem of Positive Kolmogorov-Sinai entropy for the Standard map

Oliver Knill *

This document replaces an announcement which circulated in 1999. In the present document, incorrect parts have been deleted. The entropy conjecture is open. The references given in the text might be helpful for people trying an operator theoretical or analytic approach to this problem.

Abstract

The problem of positive Kolmogorov-Sinai entropy of the Chirikov-Standard map $T_{\lambda f} : (x, y) \mapsto (2x - y + \lambda f(x), x)$ with $f(x) = \sin(x)$ with respect to the invariant Lebesgue measure on the two-dimensional is open. In 1999, we believed to have a proof that the entropy can be bounded below by $\log(\lambda/2) - C(\lambda)$ with $C(\lambda) = \operatorname{arcsinh}(1/\lambda) + \log(2/\sqrt{3})$ and that for $\lambda > \lambda_0 = (8/(6 - 3\sqrt{3}))^{1/2} = 3.1547...$, the entropy of $T_{\lambda \sin}$ should be positive. This approach was based on an idea of M. Herman using subharmonic estimates.

1 The entropy problem of the Standard map

The Chirikov-Taylor Standard map

$$T_{\lambda f} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2x - y + \lambda f(x) \\ x \end{pmatrix}$$

with $f(x) = \sin(x)$ and real parameter λ is a measure preserving diffeomorphisms on the two-dimensional torus $\mathbf{T}^2 = \mathbf{R}^2/(2\pi\mathbf{Z})^2$. It is probably the most famous example of a symplectic twist map. This map appeared in 1960 in the context of electron dynamics in microtrons (see [20]). It was first numerically studied by Taylor in 1968 and Chirikov in 1969 (these independent studies are unpublished but see [37, 19]) and also known under the name "kicked rotator" and describing ground states of the Frenkel-Kontorova model [79, 7]. It is often used to illustrate or motivate various more general mathematical theorems in smooth dynamical systems (i.e. [131]) the calculus of variations (i.e. [99, 98]) perturbation theory or renormalisation group techniques (for example to understand the break up of invariant tori [89, 135, 136]). While it is known that for $\lambda \neq 0$, the map $T_{\lambda f}$ is non-integrable, has positive topological entropy and horseshoes (i.e. [36, 4, 44]), the question, whether hyperbolicity can hold on a set of positive Lebesgue measure stays open. Similarly, while many Lebesgue measure preserving diffeomorphisms on the torus are known to be non-ergodic with positive topological entropy (i.e. [140, 157]), it is not known whether positive metric entropy is dense in the C^∞ topology. The issue of the positivity of the Lyapunov exponents on some set of positive Lebesgue measure for Hamiltonian systems has been addressed at various places or reviews [105, 157, 19, 145, 88, 91, 52, 134, 137, 16, 24, 69, 32, 155, 90, 142]. According to [113, 32], the particular mathematical problem of positive entropy of the Chirikov Standard map had been promoted in the early 80'ies by Sinai. The textbook [131] states on p. 144 a conjecture (H2) that the entropy of the Chirikov Standard map is positive for all $\lambda > 0$ and that the entropy grows to infinity for $\lambda \to \infty$.

Although numerical experiments show a very clear lower bound $\log(|\lambda|/2)$ for the entropy of the Chirikov Standard map (e.g. [37, 19, 115]), it was not known even for one single value of λ, whether the metric entropy with respect to the invariant Lebesgue measure can be positive. The second part of the (H2) conjecture of Sinai.

Conjecture 1.1 *The Kolmogorov-Sinai entropy $\mu(T_{\lambda \sin})$ of the Chirikov-Standard map $T_{\lambda \sin}$ with respect to the invariant Lebesgue measure is bounded below by* $\log(|\lambda|/2) - C(\lambda)$ *where* $C(\lambda) = \operatorname{arcsinh}(1/\lambda) + \log(2/\sqrt{3})$.

*Department of Mathematics, University of Texas, Austin, TX 78712, USA (address of 1999)

Numerical experiments suggest that one could probably even hope to get rid of the $C(\lambda)$-term in the case of the Standard map. Similar statements should hold for more general Standard maps and expect explicit lower bounds $\log(|m\lambda|/2) - C(m\lambda)$ if $\lambda \sin(x)$ is replaced by $Ex + \lambda \sin(mx)$, where the integer $E \in \mathbf{Z}$ allows additionally to tune the homotopy type of the map. This property of positive metric entropy should be stable in the real analytic category and dense in the $C^0(\mathbf{T})$ topology:

Conjecture 1.2 *There is in $C^0(\mathbf{T})$ a C^0-dense set of real-analytic Standard maps $f : \mathbf{T}^1 \mapsto \mathbf{R}$ for which $\mu(T_f) > 0$. Let f be in this dense set. In every Banach space of realanalytic maps, in which f is, there is an open neighborhood of realanalytic maps g, for which $\mu(T_g) > 0$ also.*

The conjectured sensitive dependence on initial conditions could be persistent in the realanalytic category and should be obtained by realanalytic, $C^0(\mathbf{T})$-small perturbations of integrable maps. The stability of positive metric entropy with respect to realanalytic perturbations of the map would make the result physically relevant. Other results in Hamiltonian dynamics that have both this stability and which deal with orbits forming a set of positive probability is the theory of Anosov maps (see i.e. [81, 152, 65]) (an example is $(x, y) \mapsto (4x + \lambda \sin(x) - y, x)$ for small λ) or KAM perturbation theory (see i.e. [105, 57, 67]) (which applies for example near the integrable SBKP map ([139, 12]) $(x, y) \mapsto (2x - y + 4\arg(1 + \lambda \exp(-ix)), x)$). It is not known whether there are open sets of realanalytic Hamiltonian maps and flows for which there is quasiperiodic motion on a set of positive Lebesgue measure and simultaneously a conjugation to a Markov chain on a different set of positive Lebesgue measure. [1]

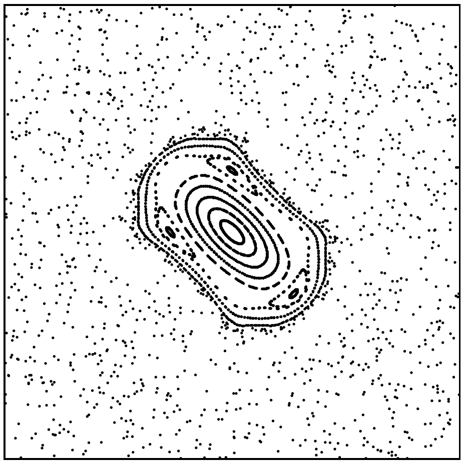
One could ask similar questions to higher-dimensional symplectic maps $(x, y) \mapsto (Ex - y + \lambda f(x), x)$ on $\mathbf{T}^{2d}$, where E is a constant symmetric matrix in $GL(d, \mathbf{Z})$ and f is a vector valued, real-analytic function on the torus. Examples are Froeschle maps, where $Ex = 2x$, $f_i(x) = \lambda_i \sin(x_i) + \sin(\sum_j x_j)$ or classes of nearest neighbor coupled map lattices [63] on $\mathbf{T}^{2d}$, where $Ex = 2x$, $f_i(x) = \lambda \sin(x_i) + \epsilon(\sin(x_{i+1} - x_i) + \sin(x_i - x_{i-1}))$ and $x_{i+d} = x_i$. All averaged Lyapunov exponents should be nonzero for large enough λ.

As for any diffeomorphism T on a compact manifold M leaving invariant a smooth measure, the entropy is by the Pesin formula [92] equal to the sum of the positive integrated Lyapunov exponents. In the two-dimensional case, the entropy is $\lim_{n \to \infty} n^{-1} \int_M \log ||dT^n(x, y)||\ dxdy$. Since an estimate of the norm of the Jacobean of the Chirikov-Standard map $T_{\lambda \sin}$ shows that the integrated Lyapunov exponent is bounded above by $\int_{\mathbf{T}^2} \log ||dT_{\lambda \sin}(x, y)||\ dxdy < \log(\lambda/2) + c(\lambda)$ with $c(\lambda) = O(1/\lambda)$, Conjecture 1.1 provides for large λ a rather accurate estimate for the actual values of the entropy of the Chirikov Standard map and shows that the measure of the Pesin region, the set on which the Lyapunov exponents is positive goes to 1, when $|\lambda| \to \infty$.

By Pesin theory [117, 94, 66, 118, 65], positive metric entropy (and in higher dimensions the non-vanishing of all Lyapunov exponents on a set of positive measure) would imply the existence of invariant sets of positive Lebesgue measure on which T_λ is ergodic and on which some iterate of T_λ is mixing and actually measure-theoretically conjugated to a Bernoulli shift. Other consequences are the density of periodic orbits in the Pesin region and shadowing properties (see [64, 65]).

An obstacle for proving positive entropy estimates for the Standard map is Donskaya's observation that elliptic islands can exist for arbitrary large λ [19, 18, 86, 32] and which make it hard to find invariant cone bundles [148, 149] in the tangent bundle. In [32] it has been shown using renormalisation of coordinates near homoclinic tangencies [104] that for a residual set of large parameters λ one has Standard maps $T_{\lambda \sin}$ which are nonergodic.

[1] There is a possibility to achieve such mixed behavior on the union of two closed symplectic manifolds, where the map is Anosov on one component and integrable on the other. For smooth, not realanalytic examples in the literature that are unstable under perturbations of the map.



Chirikov had already expressed concerns that the measure of the stable elliptic component might be close to 1 ([19] p. 333). It has been often asked whether an area-preserving monotone twist map has a dense set of elliptic islands in general (see e.g. [52, 112]). It is known that a Baire generic symplectic non-Anosov C^1-diffeomorphism has a dense set of elliptic periodic orbits [109]. Some research towards avoiding elliptic islands by smooth ergodic perturbations of the Chirikov Standard map has been done in [132]. It is possible to get positive entropy by a smooth C^1 perturbation of the map [58]. Whether this is possible with C^∞ perturbations is not known [55].

The question whether a dense set in $\mathrm{Diff}^\infty_\mu(M)$ of measure preserving diffeomorphisms on a manifold M has positive metric entropy has been asked in [59].

It has been conjectured that there exists a set of parameters λ with full density at ∞ for which the Chirikov Standard map has no elliptic islands [16] (see also [32]).

Our proof-attempts of positive metric entropy depended on spectral and complex analytic techniques as well as the determinant theory of finite von Neumann algebras avoiding the ergodicity question. We hoped to obtain many non-ergodic realanalytic Standard maps with positive metric entropy. The picture above for example shows plots of some orbits of the Chirikov Standard map in the case $\lambda = 3.4$, where a linearly stable fixed point $(1/2, 1/2)$ coexists with a region with positive metric entropy.

There are Standard maps $T_{\lambda \sin}$ arbitrarily close to this map $T_{3.4 \sin}$ for which the Birkhoff normal form at $(1/2, 1/2)$ is such that the fixed point is surrounded by invariant KAM curves of positive Lebesgue measure and for which we still have ergodic components of positive Lebesgue measure. Together with [32], we know that there exist for large λ a open dense set of parameters for the Chirikov Standard map which lead to nonergodicity and positive Kolmogorov-Sinai entropy.

Consider the matrix-valued map

$$(x,y) \in \mathbf{T}^2 \mapsto A_{E,\lambda f}(x,y) = \left(\begin{array}{cc} E + \lambda f(x) & -1 \\ 1 & 0 \end{array} \right) .$$

Together with a Lebesgue measure-preserving dynamical system $T : \mathbf{T}^2 \to \mathbf{T}^2$, it defines a cocycle $(x,y,n) \mapsto A^n_{E,\lambda f,T}(x,y) = A_{E,\lambda f}(T^{n-1}(x,y)) \circ \cdots \circ A_{E,\lambda f}(x,y)$ for which the integrated Lyapunov exponent

$$\mu(A_{E,\lambda f,T}) = \lim_{n\to\infty} \frac{1}{n} \int_{\mathbf{T}^2} \log ||A^n_{E,\lambda f,T}(x,y)|| \; dx \; dy$$

is defined. For general information on Lyapunov exponents, see for example [125, 84].

Pesin's formula links the entropy of T_f with the Lyapunov exponent of dT_f and so with the Lyapunov exponent of the cocycle because $A_{E=2,df,T_f}$ agrees with the Jacobean dT_f of the map T_f.

2 A nonselfadjoint spectral problem

Consider the random operator

$$(L_w(x,y))u_n = u_{n+1} + u_{n-1} + \lambda(w^{-1}\exp(ix_n) + w\exp(-ix_n))u_n \; , \tag{1}$$

on $l^2(\mathbf{Z},\mathbf{C})$, where $(x_n,y_n) = T^n(x,y)$ and $w \in \mathbf{C}$ is a complex parameter. For fixed $(x,y) \in \mathbf{T}^2$, this is a bounded operator on $l^2(\mathbf{Z},\mathbf{C})$. The term 'random' is used because $(x,y) \mapsto L(x,y)$ is an operator valued random variable. Such operators are in a von Neumann algebra with finite trace $\mathrm{tr}(K) = \int_{\mathbf{T}^2} [K(x,y)]_{00}\, dxdy$ (see e.g. [107, 28, 21]).

Similar as in the case, when the complex parameter is the energy, the Lyapunov exponent $\mu(w)$ of the transfer cocycle $A_E(w)$ is a subharmonic function in w. As we will see below, the Thouless formula $\mu(w) = \log(\det(L_w)) = \mathrm{tr}(\log|L_w|)$ is still true for the in general nonselfadjoint operator L_w.
The problem is to compute $\det(L_w)$. The entropy of the Standard map is $\log\det L_{w=1}$.

The average value of the Lyapunov exponent $\mu(w)$ on the circle $|w| = 1$ can be estimated with the subharmonicity argument of Herman [56] or with a Jensen formula as used by Sorets-Spencer in [133]. In higher dimensions, an adaption [48] of [133] to higher-dimensional cocycles would apply.

A theorem of Lax [82] on continuous measure-preserving transformations allows to approximate the group $\mathcal{X}$ of measure preserving homeomorphisms by finite groups $\mathcal{Y}_k$ of measure preserving transformations. If one looks at the value of the Lyapunov exponent of the matrix-valued map $A_{E,\lambda\sin,T}$ on $\mathcal{Y}_k$, (where evaluating the Lyapunov exponent is a reliable finite dimensional integration), one can observe numerically that for every transformation T, one can find a transformation $\hat{T}$ such that $\mu(A_{E,\lambda\sin,T}) + \mu(A_{E,\lambda\sin,\hat{T}}) \geq 2\log(\lambda/2)$. This relation can be understood in terms of determinants.

By looking for a proof of this relation, we were motivated by Aubry duality which can be defined if T is in such a finite group $\mathcal{Y}_k$. Aubry duality is an involutive transformation $L \mapsto \hat{L}$, which preserves the density of states such that $\log(\det(L_\lambda)) = \log(\det((\lambda/2)\hat{L}_{(4/\lambda)})) = \log(\lambda/2) + \log\det(\hat{L}_{(4/\lambda)})$. In the Mathieu case $T(x,y) = (x+y,y)$, where $L_\lambda = \hat{L}_\lambda$, this leads with $\log(\det(\hat{L}_{4/\lambda})) \geq 0$ to the estimate $\log\det(L_\lambda) \geq \log(\lambda/2)$ which is expected to be true also if T is the Standard map. Here, we will show that there exists for each transformation T and corresponding operator $L = L_{T,\lambda}$ a different transformation S and operator $\hat{L} = L_{S,\lambda}$ such that $\log(\det(L\hat{L})) \geq 2\log(\lambda/2)$.

How is the operator $\hat{L}$ obtained? The group $R_\alpha : (x,y) \mapsto (x+\alpha,y)$ of translations on the torus acts by conjugation on the whole group $\mathcal{Y}$ of measure preserving transformations. The orbits of this action are $T_\alpha = R_\alpha T R_{-\alpha}$. The operator $\hat{L}$ will be obtained by changing T to one of the conjugates T_α. The value of the Lyapunov exponent $w = \exp(i\alpha) \mapsto \mu(A_{E,\lambda f,T_\alpha})$ extends to a subharmonic function from the circle $|w| = 1$ to the entire complex plane $\mathbf{C}$. If one considers the complex parameter w as a 'spectral parameter', the Riesz measure dk used to represent the Lyapunov exponent plays the role of the density of states. It is a measure in the complex plane which has in our case in general positive area in the complex plane. The measure dk has its support in an annular neighborhood of the unit circle (and an atom at 0).

We use the artificial spectral parameter w because with respect to $z = re^{ix}$, there is no analyticity of $[z^n A^n(z,y)]_{ij}$. The motivation is trying to generalize results for maps which extend analytically to polydiscs like for $T(x,y) = (x+\alpha,y)$, where both Aubry duality [8] as well as Herman's subharmonicity [56] work directly. While the map $(te^{ix}, e^{iy}) \mapsto (te^{i(2x-y+\lambda\sin(x))}, e^{ix})$ (or any similar complexification attempt) is real-analytic in each of the variables t,x,y, it is definitely not analytic with respect to $z = te^{ix}$ and functions which needed to be subharmonic are not. Our first attack on the Standard map using plurisubharmonicity was done in the spring 1988, now more than eleven years ago. The approach was the observation that the Standard map family written in Hamiltonian form $(x,y) \mapsto (x+y+\lambda\sin(x), y+\lambda\sin(x))$ is induced on invariant tori of the single analytic map $U : (z,w,u,v) \mapsto (zwe^{z-u}, we^{z-u}, uve^{u-z}, ve^{u-z})$ on $\mathbf{C}^4$, and where nonanalyticity manifests itself that the U-invariant two-dimensional tori $S_\lambda = \{(z,w,u,v) \mid |z| = |u| = \lambda/2,\ |w| = |v| = 1\ z = \overline{u}, w = \overline{v}\}$ on which U induces T_λ, are not distinguished boundaries of polydiscs. Later attempts were to Vlasov-Toda deform the cocycle with the aim to estimate the Lyapunov exponent on tori of isospectrally deformed operators (on which the Lyapunov exponent is constant) [71, 70] or to fix the dynamical system and to vary the cocycle [69].

Because $w \mapsto g_n(w,x,y) = w^n[A_E^n(w,x,y)]_{11}$ is analytic, the $r = |w|$ dependence of the local Lyapunov exponent can be related with the average angular dependence of the argument: in polar coordinates $w = re^{i\phi}$,

the Cauchy-Riemann differential equations

$$\frac{d}{d\phi}\frac{1}{r}\int_{\mathbf{T}^2} \arg(g_n(w,x,y))\; dxdy = \frac{d}{dr}\int_{\mathbf{T}^2} \log|g_n(w,x,y)|\; dxdy$$

are valid in any connected region of the w- resolvent set. In the complement, in the support of the w-spectrum, this formula can not be used and an integrated version, a Jensen formula replaces it. Besides the contribution of the radial change of the argument, there is an additional nonnegative subharmonic contribution to the Lyapunov exponent.

3 More heuristics

Let $A_{E,\lambda\cos}(z,y) = \begin{pmatrix} E + \frac{\lambda}{2}(z+z^{-1}) & -1 \\ 1 & 0 \end{pmatrix}$.

In situations like the Mathieu case $T(x,y) = (x+y,y)$, where T extends to an analytic map on $\mathbf{D}\times\mathbf{T}$ with respect to $z = r\exp(ix)$, Herman's subharmonicity [56] gives $\mu(A_T) - \log(\lambda/2) \geq 0$. For a general map $T \in \mathcal{Y}$ which does no more commute with $R_\alpha : (x,y) \mapsto (x+\alpha,y)$ on $\mathbf{T}^2$, the function $w = \exp(i\alpha) \mapsto \mu(A_{E,\lambda\sin,T_\alpha})$ with $T_\alpha = R_\alpha T R_{-\alpha}$ is no more constant. In average, we have $\int_{\mathbf{T}} \mu(A_{E,\lambda\sin,T_\alpha})\; d\alpha \geq \log(\lambda/2)$. Furthermore $\mu(A_{E,\lambda\sin,T_\alpha})$ is bounded above by $\log(\lambda/2) + c(\lambda)$ with $c(\lambda) = O(1/\lambda)$. This means that for large λ, we have $\mu(A_{E,\lambda\sin,T_\alpha}) \geq \log(\lambda/2)$ for a large set of α's. We used such a fact in a similar way in [69].

However, we don't know the specific value of the upper-continuous function $\alpha \mapsto \mu(A_{E,\lambda\sin,T_\alpha})$ at the point $\alpha = 0$, which we are interested in. It could be zero a priori. Numerical computations however show that the minimum of $\alpha \mapsto \mu(A_{E,\lambda\sin,T_\alpha})$ is not farther away from the mean value than the maximum. There is a heuristic explanation which uses the Jensen formula in a sector and which will assume $T \in \mathcal{X}$. This Jensen formula (see Section 9) is essentially an integrated Cauchy-Riemann differential equation and sharpens the subharmonicity tool. We also use the Lax approximation theorem in Section 10 to explain the now following heuristics.

The Lyapunov exponent of the matrix-valued map $(x,y) \in \mathbf{T}^2 \mapsto B(e^{ix})$ with

$$B(z) = zA_{E,\lambda\cos}(z) = z\begin{pmatrix} E + \frac{\lambda}{2}(z+z^{-1}) & -1 \\ 1 & 0 \end{pmatrix}$$

is the same as the Lyapunov exponent of the cocycle $(x,y) \mapsto A_{E,\lambda\cos}(x)$ because $|z| = 1$.

Define for $(x,y) \in \mathbf{T}^2$ and $n \in \mathbf{N}$ the complex function

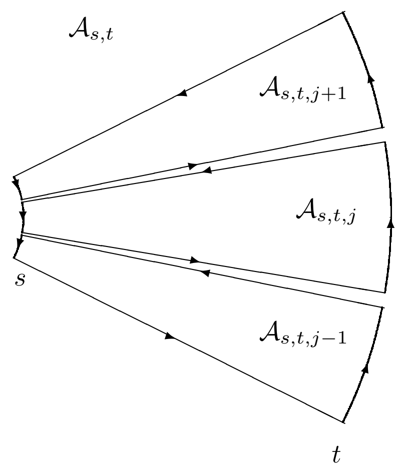
$$w \mapsto g_{n,T}(w^{-1}\exp(ix),y) = [B^n_{E,\lambda\cos,T}(w^{-1}\exp(ix),y)]_{11}$$

which is analytic in the complex plane.

Denote by $\mathrm{Arg}_{[z_1,z_2]}(g_{n,T})$ the argument change of the analytic function $g_{n,T}$ on the line from z_1 to z_2. Unlike $\arg(g_{n,T})$, the argument change $\mathrm{Arg}_{[z_1,z_2]}(g_{n,T})$ is a uniquely defined number.

Let T_k be a cyclic Lax cube exchange transformation approximating a homeomorphism $T \in \mathcal{X}$. The map T_k is extends from $\mathbf{T}^2$ to $\mathbf{D}\times\mathbf{T}$ by setting $T_k(r\exp(ix),y) = (r\exp(ix_1),y_1)$ with $T_k(x,y) = (x_1,y_1)$.

In each sector, we could also take the z variable instead of the w variable because $z \mapsto A^n(z,y)$ is analytic in each sector $\mathcal{A}_{s,t,j} = \{w \in \mathbf{C} \mid |w| \in [s,t], \arg(w) \in [2\pi j/k, 2\pi(j+1)/k)\ \}$.



Let $\Gamma^+(\alpha)$ be path connecting $se^{i\alpha}$ and $te^{i\alpha}$ on a radial line segment, Γ^- the reversed path. Let $\gamma_s^+(\alpha, \beta) = \{w, |w| = s, \arg(w) \in [\alpha, \beta]\}$ be a circular arc contained in the circle $\gamma_s = \{|w| = s\}$ and $\gamma_s^-(\alpha, \beta)$ the reversed path. The difference between the argument change of the function $g_{n,T}(w) = [B^n_{E,\lambda \sin,T}(w)]_{11}$ along the circles γ_s^- and γ_t^+ enclosing the annulus $\mathcal{A}_{s,t} = \{|w| \in [s,t]\}$ is the sum of the k angular argument changes $\mathrm{Arg}_{\gamma_t^+(2\pi j/k, 2\pi(j+1)/k)}(g_{n,T}) - \mathrm{Arg}_{\gamma_s^-(2\pi j/k, 2\pi(j+1)/k)}(g_{n,T})$ plus the sum of the k radial argument changes $\mathrm{Arg}_{\Gamma^+(2\pi j/k)}(g_{n,T}) - \mathrm{Arg}_{\Gamma^-(2\pi(j+1)/k)}(g_{n,T})$ with $j = 1, \dots, k$.

The total sum of the radial plus angular argument changes is nonnegative because it is the sum of the indices of the roots of $z \mapsto g_{n,T}$ inside the annulus $\mathcal{A}_{s,t}$, all of which are nonnegative.

The sum of the k radial argument changes is bounded above by a $O(1/\lambda)$-term, because a positive value contributes positively to the Lyapunov exponent estimate and the Lyapunov exponent is bounded above by $\log(\lambda/2) + c(\lambda)$ with $c(\lambda) = O(1/\lambda)$. There is no reason, why the sum of the k radial argument changes should differ much from its negative. Numerical experiments strongly confirmed that they don't. If this sum of the radial argument changes

$$\sum_{j=1}^{k} \mathrm{Arg}_{\Gamma^+(2\pi j/k)}(g_{n,T}) - \mathrm{Arg}_{\Gamma^-(2\pi(j+1)/k)}(g_{n,T}) \tag{2}$$

(which is averaged over $\mathbf{T}^2$ and bounded uniformly in k) would vanish, one would have the same estimates as in the subharmonic case and

$$n^{-1} \int_{\mathbf{T}^2} \log |g_{n,T}(x,y,w)| \, dxdy \geq n^{-1} \int_{\mathbf{T}^2} \log |g_{n,T}(x,y,0)| \, dxdy = \log(\lambda/2)$$

for all $|w| = 1$. This would imply $\mu(A_{E,\lambda \cos,T}) \geq \log(\lambda/2)$.

While the sum of the radial argument changes (2) does not vanish for a general $T \in \mathcal{Y}$, numerical data makes (2) appear to be small. Indeed, we will see that for every transformation $T \in \mathcal{Y}$, there is an other transformation $S_\alpha = R_\alpha T R_{-\alpha}$ such that the Lyapunov exponent of A_T plus the Lyapunov exponent of A_S beats $2\log(\lambda/2)$. This will imply that the fluctuations of

$$\alpha \mapsto \mu_n(\alpha) = n^{-1} \int_{\mathbf{T}^2} \log |[A^n(e^{i(\alpha+x)}, y)]_{11}| \, dxdy$$

around its mean $\int_{\mathbf{T}} \mu_n(\alpha) \, d\alpha \geq \log(\lambda/2)$ can not get too large.

We have not been able to prove such a statement using complex analytic methods only.

4 The spectrum of periodic difference operators

This section revies some facts about not necessarily selfadjoint periodic Jacobi matrices and higher order difference operators. Unlike in the selfadjoint case, when the spectrum is confined to the real axes, the topology of the spectrum can then be more interesting. All we need to know for our purposes is that the spectrum is contained in a nowhere dense set.

While nonselfadjoint differential operators have been studied quite a bit by Russian mathematicians, nonselfadjoint higher order difference operators have appeared less frequently in the literature. But they do occur: these

operators appear naturally when studying the stability of stationary solutions in coupled nonharmonic oscillators [144, 62]. For nonselfadjoint Schrödinger differential operators, see [45] and references.

Jacobi matrices. Three vectors $\underline{a}, \underline{b}, \underline{c} \in \mathbf{C}^p$ define periodic sequences $a_n = a_{n+p}, b_n = b_{n+p}, c_n = c_{n+p} \in l^\infty(\mathbf{Z}, \mathbf{C})$ which can be used to build a p-periodic Jacobi matrix

$$(Lu)_n = a_n u_{n+1} + c_{n-1} u_{n-1} + b_n u_n$$

which is a bounded operator on on $l^2(\mathbf{Z}, \mathbf{C})$. Denote by L_{per} the same operator L but acting on a different Hilbert space, namely the finite dimensional space of p-periodic sequences equipped with the norm $(\sum_{i=1}^p |u_i|^2)^{1/2}$. For $w \in \mathbf{C}$, (a parameter which has nothing to do with the complex parameter w in the operator (1)), this operator is given by the matrix

$$L_{per}(w) = \begin{pmatrix} b_1 & a_1 & 0 & \cdot & 0 & w^{-1} c_p \\ c_1 & b_2 & a_2 & \cdot & \cdot & 0 \\ 0 & c_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_{p-2} & 0 \\ 0 & \cdot & \cdot & c_{p-2} & b_{p-1} & a_{p-1} \\ w a_p & 0 & \cdot & 0 & c_{p-1} & b_p \end{pmatrix}$$

and satisfies $L_{per} = L_{per}(1)$. Denote by $\Delta(z, w) = \det(L_{per}(w) - z)$ the characteristic polynomial of $L_{per}(w)$ and by $\Delta(z) = \Delta(z, 1)$ the characteristic polynomial of L_{per}. Comparing coefficients in w and using $\Delta(z, 1) = \Delta(z)$, one gets

$$\Delta(z, w) = \Delta(z) - aw - cw^{-1} - b$$

with $a = \prod_{j=1}^p a_j$ and $c = \prod_{j=1}^p c_j$ and $b = -a - c$. Denote by τ the shift $u_n \mapsto u_{n+1}$ on $l^2(\mathbf{Z}, \mathbf{C})$.

By the theorem of Cayley-Hamilton, we have $\Delta(L_{per}) = 0$. We deduce from this that the operator $\Delta(L, \tau^p)$ on $l^2(\mathbf{Z}, \mathbf{C})$ defined by the functional calculus (plug-in the operator L for z and the operator τ^p for w in $\Delta(z, w)$) satisfies

$$0 = \Delta(L_{per}) = \Delta(L, \tau^p) = \Delta(L) - a\tau^p - c\tau^{-p} - b\ .$$

This implies that the operator $K := \Delta(L) = a\tau^p + c(\tau^p)^* + b$ on $l^2(\mathbf{Z}, \mathbf{C})$ is a Laplacian with space independent entries. Conjugated by a Fourier transform $U : l^2(\mathbf{Z}, \mathbf{C}) \to L^2(\mathbf{T}, \mathbf{C})$, it is unitarily equivalent to the multiplication operator

$$(UKU^*)u(\theta) = (a\, e^{ip\theta} + b + c\, e^{-ip\theta})u(\theta) =: \lambda(\theta)u(\theta)$$

on $L^2(\mathbf{Z}, \mathbf{C})$ which has as the spectrum on an ellipse $\mathcal{E}$ possibly being degenerated to an interval. Since $\Delta(L) = K$, we know by the spectral theorem that $\sigma(\Delta(L)) = \Delta(\sigma(K))$. The spectrum of L consists of p closed real curves. They are the zeros of $z \mapsto \Delta(z) - w$, when w is runs through the ellipse $\mathcal{E}$. The density of states of L is the push-forward measure $(\Delta^{-1} \circ \lambda)^* d\theta$ of the Lebesgue measure $d\theta$ on the circle $\mathbf{T} = \mathbf{R}/(2\pi\mathbf{Z})$ under the multi-valued map $\phi = \Delta^{-1} \circ \lambda : \mathbf{T} \mapsto \mathbf{C}$. (See [46] for a derivation in the case $a_n = c_n = 1$).

Remarks.
1) Since the spectrum of Jacobi matrices is simple, the curves $z_i(\theta)$ do not intersect transversely. However, by deforming along a one-parameter family of Jacobi matrices, curves could merge or separate. The discriminant $\{L \mid (\Delta_L)'(z) = 0 \text{ for some } z \in \sigma(L)\}$ is the set of Jacobi matrices, where the topology of the spectrum can change under small parameter variations by merging or separation of spectral arcs.

2) The Bloch variety of L is the set $BV(L) = \{(z, w) \in \mathbf{C}^* \times \mathbf{C}^* \mid \Delta(z) = aw + b + cw^{-1}\}$, where $\mathbf{C}^*$ is the Riemann sphere $\mathbf{C} \cup \{\infty\}$. It is the set of all (z, w) such that there exists a nontrivial solution of $Lu = zu$ satisfying $(\tau^p u)_n = u_{n+p} = wu_n$. The spectrum of L is the intersection of $BV(L)$ with $\{|w| = 1\}$.

3) A theorem of Polya (see [1]) implies that the projection of the spectrum of a periodic (not necessarily selfadjoint) Jacobi matrix onto any real line in the complex plane has Lebesgue measure ≤ 4.

4) In the case when a_j, b_j, c_j are real and $a_n = c_n$ and $a = \prod_j a_j = 1$, the Jacobi matrix L is real and selfadjoint and K is the free Laplacian $\tau + \tau^* - 2$ with spectrum $[-2, 2]$ so that $\sigma(L) = \{z \mid \Delta(z) \in [-2, 2]\}$. The spectrum is then a union of bands on the real line which can intersect only at the boundaries.

5) In the case when $a = \prod_j a_j = 0, c = \prod_j b_j = 0$, the spectrum is $\Delta^{-1}(0)$, the set of roots of the characteristic polynomial of L_{per} confirming that in this case, L and L_{per} have the same spectrum. (L is then a countable direct sum of operators L_{per}.)

Higher order difference operators. For a p-periodic operator $L = \sum_{i=-r}^{r} a_i \tau^i$, the spectrum of L_{per} is more complex. The spectral bands curves can cross and for selfadjoint matrices, the bands can overlap.

We consider the case $p = kN$, where the operator can be rewritten as a Jacobi operator $L = a\tau + c\tau^* + b$ on $l^2(\mathbf{Z}, \mathbf{C}^N)$, where a_j, b_j, c_j are $N \times N$ matrices satisfying $a_{j+k} = a_j, b_{j+k} = b_j, c_{j+k} = c_j$.

Lemma 4.1 (Spectrum of periodic matrix-valued Jacobi matrices) *The spectrum of the Jacobi operator L with $N \times N$-matrix valued coefficients is contained in the union of at most $p = kN$ closed real curves.*

Proof. Unlike in the case $N = 1$, the Bloch variety

$$\{(z, w) = \Delta(z, w) = \det(L_{per}(w) - z) = 0\} \subset \mathbf{C}^2$$

can in general no more given explicitly in the form $f(w) = g(z)$. The spectrum is still the intersection of this complex variety with $\{|w| = 1\}$, the parameter $|w| = 1$ labeling Bloch waves.

The intersection of the Bloch variety with $|w| = 1$ is a set in the complex plane which is contained in a union of at most $p = kN$ closed curves.

Let Det denote the determinant function on the algebra of complex $N \times N$ matrices. We write $\prod_{j=1}^{k} a_j = a_k \dots a_1$, also in the noncommutative case. With $a = \mathrm{Det} \prod_{j=1}^{k} a_j, c = \mathrm{Det} \prod_{j=1}^{k} c_j$, we have

$$\Delta(z, w) = \Delta(z) + aw^N + cw^{-N} + f(z, w) ,$$

where the mixed term $f(z, w)$ is a polynomial in w, w^{-1}, z which is of degree $< N$ in w, w^{-1} and of degree $< p$ in z.

Examples.
1) We are especially interested in higher order difference operators which arethe product $L = L^{(1)} L^{(2)}$ of two Jacobi matrices $L^{(i)} = \tau_i + \tau_i^* + V^{(i)}$.

2) If a_n, b_n, c_n are diagonal, the operator L is the direct sum of N one-dimensional Jacobi matrices. There are therefore operators with $p = kN$ spectral curves.

5 Thouless formula for difference operators

This section reviews the Thouless formula for operators on the strip, that is Jacobi matrices with matrix-valued entries. We recall the proof in [80] because the class of operators differs a tiny bit from the operators considered in [80]: the off diagonal matrices are the identity to which some nilpotent part is added. An other change is that we do not assume selfadjointness of the operator. The proof of the Thouless formula however is the same.

The Thouless formula $\mu(A_E) = \int \log |E - E'| \, dk(E')$ relates the density of states dk of a random Jacobi operator $\tau + \tau^* + b$ with the Lyapunov exponent $\mu(A_E)$ of the transfer cocycle (see [23, 17]). This generalizes to operators $\tau + \tau^* + b$ on $l^2(\mathbf{Z}, \mathbf{C}^N)$, where $\tau u_n = u_{n+1}$ and $b(x)$ is a selfadjoint $N \times N$ matrix, if $\mu(A_E)$ is the average of the N largest Lyapunov exponent of the $2N \times 2N$-cocycle [80]. The Thouless formula extends also to the case of operators $a\tau + (a\tau)^* + b$, where $a(x, y)$ are unitary [74].
The next lemma provides a generalization to operators of the type $L = L^{(1)} L^{(2)}$ with Jacobi matrices

$$(L^{(k)}(x^{(k)}, y^{(k)})u)_n = u_{n+1} + u_{n-1} + V_n^{(k)}(x^{(k)}, y^{(k)})u_n ,$$

where the later are not necessarily selfadjoint operators on $l^2(\mathbf{Z},\mathbf{C})$,

$$L(x^{(1)},y^{(1)},x^{(2)},y^{(2)}) = \begin{pmatrix} \cdot & \cdot & \cdot & 1 & 0 & 0 & 0 \\ 1 & c_{n-2} & b_{n-1} & a_{n-1} & 1 & 0 & 0 \\ 0 & 1 & c_{n-1} & b_n & a_n & 1 & 0 \\ 0 & 0 & 1 & c_n & b_{n+1} & a_{n+1} & 1 \\ 0 & 0 & 0 & 1 & c_{n+1} & b_{n+2} & \cdot \\ 0 & 0 & 0 & 0 & 1 & \cdot & \cdot \end{pmatrix} , \tag{3}$$

where $a_n = a_n(\mathbf{x}) = V_n^{(1)}(x^{(1)},y^{(1)}) + V_{n+1}^{(2)}(x^{(2)},y^{(2)}), b_n = b_n(\mathbf{x}) = V_n^{(1)}(x^{(1)},y^{(1)})V_n^{(2)}(x^{(2)},y^{(2)}) + 2, c_n = c_n(\mathbf{x}) = V_n^{(1)}(x^{(1)},y^{(1)}) + V_{n-1}^{(2)}(x^{(2)},y^{(2)})$. The potentials $V_n^{(j)}(x^{(j)},y^{(j)})$ are obtained from $T_j \in \mathcal{Y}$ and functions $V^{(j)} : \mathbf{T}^2 \mapsto \mathbf{R}$ by $V_n^{(j)}(x^{(j)},y^{(j)}) = V^{(j)}(T_j^n(x^{(j)},y^{(j)}))$.

Each difference operator operator $L(\mathbf{x})$ on $l^2(\mathbf{Z},\mathbf{C})$ can be written as a Jacobi operator

$$\tilde{L} = \tilde{a}\tilde{\tau} + (\tilde{c}\tilde{\tau})^* + \tilde{b}, (\tilde{L}u)_{\tilde{n}} = \tilde{a}_{\tilde{n}}u_{\tilde{n}+1} + \tilde{c}_{\tilde{n}-2}u_{\tilde{n}-1} + \tilde{b}_{\tilde{n}}u_{\tilde{n}} \tag{4}$$

on $l^2(\mathbf{Z},\mathbf{C}^2)$ with $\tilde{\tau}u_{\tilde{n}} = u_{\tilde{n}+1}, \tilde{\tau}\tilde{c}(\mathbf{x}) = \tilde{c}(T_1^2(x^{(1)},y^{(1)}),T_2^2(x^{(2)},y^{(2)}))\tilde{\tau}$ and with matrix-valued entries

$$\tilde{b}_{\tilde{n}} = \begin{pmatrix} b_{n-1} & a_{n-1} \\ c_{n-1} & b_n \end{pmatrix}, \tilde{a}_{\tilde{n}} = \begin{pmatrix} 1 & 0 \\ a_n & 1 \end{pmatrix}, \tilde{c}_{\tilde{n}} = \begin{pmatrix} 1 & c_n \\ 0 & 1 \end{pmatrix} ,$$

where $n = 2\tilde{n}$. The transfer cocycle

$$A_E = \begin{pmatrix} (E-\tilde{b}_{\tilde{n}})\tilde{a}_{\tilde{n}-2}^{-1} & -\tilde{c}_{\tilde{n}-2} \\ \tilde{a}_{\tilde{n}-2}^{-1} & 0 \end{pmatrix} = \begin{pmatrix} E-b_{n-1}+a_{n-1}a_{n-2} & -a_{n-1} & -1 & -c_{n-2} \\ a_{n-2}(b_n-E)-c_{n-1} & E-b_n & 0 & -1 \\ 1 & 0 & 0 & 0 \\ -a_{n-2} & 1 & 0 & 0 \end{pmatrix} \tag{5}$$

satisfies

$$A_E \begin{pmatrix} \tilde{a}_{n-2}u_n \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} \tilde{a}_n u_{n+1} \\ u_n \end{pmatrix}$$

if $(\tilde{L}(\mathbf{x})u)_n = Eu_n$ for $u \in l^2(\mathbf{Z},\mathbf{C}^2)$. The density of states of L defined as the functional $f \mapsto \int_{\mathbf{T}^4} [f(L(\mathbf{x})]_{00}\, d\mathbf{x}$ on $C(\mathbf{C})$ and the density of states of $\tilde{L}$ defined as the functional $f \mapsto \int_{\mathbf{T}^2} \mathrm{Trace}([f(\tilde{L}(\mathbf{x})]_{00})\ \mathbf{x}$ both exist and they are the same probability measure in $\mathbf{C}$, if Trace is the normalized trace on 2×2 matrices satisfying $\mathrm{Trace}(1) = 1$.

Remarks.
1) For $L = L^{(1)}L^{(2)}$, where $L^{(i)}$ are Jacobi operators, there are other transfer matrices like the product of the individual 2×2 transfer matrices of the operators $L^{(i)}$. These are different matrix-valued functions in E and the Thouless formula would look different in this case.

2) Note that dk is normalized to be a probability measure. In some of the literature like for example in [80], the usual Trace for $N \times N$ matrices is taken which implies that the total mass of the density of states dk for operators L on the strip becomes N.

Lemma 5.1 (Thouless formula for operators on the strip) *Let dk be the density of states of the operator (4). Then*

$$\int_{\mathbf{C}} \log|E-E'|\, dk(E') = \mu(\wedge^2 A_E)/2 ,$$

where $\mu(\wedge^2 A_E)/2$ is the arithmetic mean of the two largest Lyapunov exponents of the 4×4 transfer cocycle A_E of the operator (4).

Proof. Small changes are needed to the known proofs [23, 17, 80] for the same statements for selfadjoint operators L, where $a_n = c_n = 1, n \in \mathbf{Z}$. Write $\mathbf{x} = (x^{(1)},y^{(1)},x^{(2)},y^{(2)})$ for a point on $\mathbf{T}^4$.

(i) It is enough to prove the statement for periodic, (nonergodic) transformations $T \times S$.
Proof. Every $T \times S \in \mathcal{Y}\times\mathcal{Y}$ can be approximated by periodic transformations $T_n \times S_n$ using Rohlin's lemma

[53] (which implies that there exist measurable sets Y_n whose Lebesgue measure on $\mathbf{T}^4$ goes to 1 for $n \in \infty$ so that $(T \times S)^k(\mathbf{x}) = (T_n \times S_n)^k(\mathbf{x})$ for $k = 0, \dots n-1$).

The Thouless formula is an identity for subharmonic functions $f_{T\times S} = g_{T\times S}$, parameterized by $T \times S \in \mathcal{Y} \times \mathcal{Y}$. Let $dk(f)$ denote the Riesz measure of a subharmonic function f it satisfies $dk(f) = \Delta f$ in the sense of distributions, where Δ is the Laplacian. It follows from the Avron-Simon lemma and the dominated convergence theorem that if $T_n \times S_n \to T \times S$ in the uniform topology for measure preserving transformations, then $dk(f_{T_n \times S_n}) \to dk(f_{T\times S})$ weakly. Also $dk(g_{T_n\times S_n}) \to dk(g_{T\times S})$ weakly holds because $dk(g_T)$ is a weak limit of measures $dk(g_T^{(k)}) = \int_{\mathbf{T}^4} \sum_j \delta(E_j^{(k)}(\mathbf{x}))\, d\mathbf{x}$, where $E_j^{(k)}(\mathbf{x})$ are the roots of $E \mapsto [A^k_{T,E}(\mathbf{x})]_{11}$ and $T \mapsto dk(g_T^{(k)})$ is continuous from $\mathcal{Y}$ with the uniform topology to the space of measures on $\mathbf{C}$ with the weak topology.
By assumption, we have $dk(f_{T_n\times S_n}) = dk(g_{T_n\times S_n})$. Therefore $dk(f_{T\times S}) = dk(g_{T\times S})$ which is $\Delta f_{T\times S} = \Delta g_{T\times S}$. From this, $f_{T\times S} = g_{T\times S}$ follows by Weyl's lemma [121].

(ii) Next, it is sufficient to assume that E is outside the spectrum of a periodic operator L (which is now contained in a finite union of one-dimensional curves (Lemma (4.1))) as well as that E is outside the spectrum of $L^{(0)}$, where $L^{(0)}$ is the Jacobi matrix on $l(\mathbf{Z}^+, \mathbf{C})$ with zero Dirichlet boundary condition at $n = 0$. The spectrum of $L^{(0)}$ is a finite set of points.
Proof. This is the Craig-Simon subharmonicity argument [22]. Taking balls $B_r(E)$ around a point E in the spectrum, the result holds for the over $B_r(E)$ averaged Lyapunov exponent. Passing to the limit $r \to 0$ is possible because of subharmonicity.

(iii) If $\lambda_j(E) = \prod_{k=1}^n (E - \lambda_{jk}), j = 1, \dots, N$ is an eigenvalue of the $N \times N$ matrix $[A_E^n]_{11}$, (where the $2N \times 2N$ matrix A_E is written as a 2×2 matrix of $N \times N$ matrices), then each root λ_{jk} of $E \mapsto \lambda_j(E)$ gives rise to an eigenvector $u^{(jk)}$ of a truncation $L^{(n)}$ of L, where a_k, b_k, c_k are put to zero for $k < 0$ as well as for $k > n$. The measures $dk_n = (2n)^{-1} \sum_{j,k} \delta(\lambda_{jk})$ converge for $n \to \infty$ weakly to the density of states dk which is supported on finitely many curves.
This is the Thouless-Avron-Simon lemma [23] which holds in general for random finite-difference operators which do not need to be selfadjoint.

(iv) The fact that

$$\int_{\mathbf{C}} \log|E - E'|\, dk_n(\mathbf{x}, E') = \sum_{j,k} \log|E - \lambda_{jk}(\mathbf{x})| = \log|\mathrm{Det}([A_E^n]_{11}(\mathbf{x}))|$$

converges to $\mu(\wedge^2 A_E(\mathbf{x}))/2$ for almost all $\mathbf{x} \in \mathbf{T}^4$ in the limit $n \to \infty$ is seen in the same way as in the Appendix of [80]: If $m^\pm(\mathbf{x}) = a(\mathbf{x}) u^\pm((T \times S)\mathbf{x}) u^\pm(\mathbf{x})^{-1}$ are the matrix-valued Titchmarsh-Weyl functions, where $Lu_n^\pm(\mathbf{x}) = Eu_n^\pm(\mathbf{x})$ and $\{u_n^\pm(\mathbf{x})\}_{n\in\pm\mathbf{Z}^+} \in l^2(\pm\mathbf{Z}^+, M(N, \mathbf{C}))$, then

$$(\mp 1/2) \int_{\mathbf{T}^4} \log|\mathrm{Det}(m^\pm(\mathbf{x}))|\, d\mathbf{x}$$

is the arithmetic mean $\mu(\wedge^2 A_E(\mathbf{x})/2$ of the two largest Lyapunov exponents of A_E. Furthermore, the maximal Lyapunov exponent averaged over $\mathbf{T}^4$ is

$$-\int_{\mathbf{T}^4} \log||m^+(\mathbf{x})||\, d\mathbf{x} = \int_{\mathbf{T}^4} \log||m^-(\mathbf{x})||\, d\mathbf{x}\ .$$

Because E is not an eigenvalue of $L(\mathbf{x})$, there are two $N \times N$-matrices U, V such that the functions $m^\pm, m^{(0)} = au^{(0)}(T \times S)(u^{(0)})^{-1}$ satisfy $m^{(0)} = m^- U + m^+ V$. Here $u^{(0)}(\mathbf{x})$ is an other solution of $L(\mathbf{x})u = Eu$ satisfying a Dirichlet boundary condition at 0. The matrix U is invertible because else there would exists $a \neq 0$ with $Ua = 0$ and $m^{(0)}a = m^+ Va$ would be in $l^2(\mathbf{Z}^+, \mathbf{C}^2)$ and contradict the fact that E was also chosen away from the spectrum of $L^{(0)}$. (In the selfadjoint case, this would already have been taken care of by choosing $\mathrm{Im}(E) \neq 0$.)
Writing

$$m_n^{(0)} = (1 + m_n^+ V U^{-1} (m_n^-)^{-1}) m_n^- U$$

and using $m_n^{(0)}(\mathbf{x}) = [A_E^n]_{11}(\mathbf{x})$, one concludes

$$\log\det(L) \quad = \quad \int_{\mathbf{C}} \log|E - E'|\, dk(E')$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \int_{\mathbf{T}^4} \int_{\mathbf{C}} \log|E-E'| \, dk_n(\mathbf{x},E') \\
&= \lim_{n\to\infty} \int_{\mathbf{T}^4} \log \mathrm{Det}([A_E^n(\mathbf{x})]_{11}) \, d\mathbf{x} \\
&= \frac{1}{2} \lim_{n\to\infty} \int_{\mathbf{T}^4} \log|\mathrm{Det}(m^{(0)}(\mathbf{x})_n)| \, d\mathbf{x} \\
&= \frac{1}{2} \int_{\mathbf{T}^4} \log|\mathrm{Det}(m^{-}(\mathbf{x}))| \, d\mathbf{x} = \mu(\wedge^2 A_E)/2 \; .
\end{aligned}
$$

6 An upper bound for the Lyapunov exponent

In this section we determine the constant $C(\lambda)$ in Conjecture (1.1) and formulate a Proposition which outlines, how one can improve the constant with more effort. The reason to stick to the constant $C(\lambda)$ is that it allows explicit expressions. We also show in this section that $C(\lambda)$ can be replaced by $c(\lambda) = O(\lambda^{-1})$.

Because $||A^n|| \leq ||A||^n$ for any matrix A, the number

$$m(E,\lambda) := \int \log ||A_{E,\lambda\cos}(x,y)|| \; dxdy$$

is an upper bound for the Lyapunov exponent of the Standard map, rsp. the Lyapunov exponent of the cocycle $A_E(w)$ for $|w| = 1$. Define $C(E,\lambda) = m(E,\lambda) - \log(\lambda/2)$ and $C(\lambda) = C(0,\lambda)$.

Lemma 6.1 *$C(E,\lambda) = \log(2/\sqrt{3}) + O(1/\lambda)$ for large λ and $C(0,\lambda) > 0$ for $\lambda > \lambda_0 = 2\sqrt{2/(6-3\sqrt{3})} =$ 3.1547....*

Proof. With the norm $|||A||| = \max_{i=1,2} |Ae_i|$, where e_i are the basis vectors, there is an explicit expression for an upper bound of the integral $m(E,\lambda)$. Following [60], define

$$M(E,\lambda) = |E + i + \sqrt{(E+i)^2 - \lambda^2}|/\sqrt{3}$$

where the square-root takes the solution with positive imaginary part.

For matrices $A = \begin{pmatrix} c & 1 \\ -1 & 0 \end{pmatrix}$ one has $|||A||| = |\sqrt{1+c^2}| \geq (\sqrt{3}/2)||A||$. Furthermore, with the analytic function $f(z) = z^2(1 + (E - \lambda(z+z^{-1})/2)^2) = \lambda/2(z-a_-)(z-a_+)$ where $|a_-| < 1$, one has using the Jensen formula

$$
\begin{aligned}
\int_{\mathbf{T}} \log|||A_{E,\lambda\cos}(x)||| \; dx &= \frac{1}{2}\int_{\mathbf{T}} \log|1 + (E - \lambda\cos(x))^2| \; dx \\
&= \frac{1}{2}\int_{\{|z|=|\exp(ix)|=1\}} \log|f(z)| \; dx \\
&= \frac{1}{2}(\log(f(0)) - \log(|a_-|)) \\
&= \log(\frac{\sqrt{3}}{2} M(E,\lambda)) \; .
\end{aligned}
$$

Therefore $C(E,\lambda) = \log(M(E,\lambda)) - \log(\lambda/2) = \log(2/\sqrt{3}) + \log(|(E+i)/\lambda + \sqrt{(E+i)^2/\lambda^2 - 1}|)$ and $C(0,\lambda) = C(\lambda) = \log(1/\lambda + \sqrt{1/\lambda^2 + 1}) + \log(2/\sqrt{3}) = \mathrm{arcsinh}(1/\lambda) + \log(2/\sqrt{3})$. For $\lambda = 3.1547.. = 2\sqrt{2/(6-3\sqrt{3})}$, one has $C(\lambda) = 0$.

The function $\lambda \mapsto C(\lambda)$, having a positive derivative, is strictly monotone.

Lemma 6.2 *We have*

$$\int_{\mathbf{T}^2} \log ||dT_{E,\lambda \cos}(x,y)|| \; dxdy \leq c(E,\lambda) := \int_{\mathbf{T}} \log \sqrt{2 + (E + \lambda \cos(x))} \; dx = O(\lambda^{-1}) \; .$$

Proof. We use Hadamard's determinant theorem, which tells that for a finite matrix L, one has $\log \det(L) \leq \sum_{j=1}^n ||a_j||_2$, where a_j are the rows of the matrix $L = (a_1, \cdots, a_n)$. (The relation follows from the fact that if $A = (a_1, \dots, a_n)$, then $\det(A)/\prod_{j=1}^n ||a_j||_2 = \det(B)$, where $B = (a_1/||a_1||, \dots, a_n/||a_n||)$. The determinant theorem is equivalent to $\det(B) \leq 1$, a relation which can easily be seen geometrically (i.e. [9, 29]).)

In the random case, where the a_j are vector valued random variables on $\mathbf{T}^2$ with Lebesgue measure, and $L^{(n)}$ is a finite dimensional $n \times n$ approximation of L and where $n^{-1} \log \det(L^{(n)})$ converges to to the Fuglede-Kadison determinant $\log \det(L)$, we obtain from Hadamard's determinant inequality and Birkhoff's ergodic theorem

$$\log \det(L) \leq \int_{\mathbf{T}^2} \log ||a(x,y)|| \; dxdy \; .$$

In the case of random Jacobi operators $L = \tau + \tau^* + b(x) = \tau + \tau^* + \lambda \cos(x) + E$, where $a(x,y) = (\dots, 0, 0, 1, b(x), 1, 0, \dots)$, this gives

$$\begin{aligned} \log \det(L) \leq \int_{\mathbf{T}} \log \sqrt{2 + (E + \lambda \cos(x))^2} \; dx &= \log(\lambda/2) + \int_{\mathbf{T}} \log \sqrt{8/\lambda^2 + 4(E/\lambda + \cos(x))^2} \; dx \; , \\ &= \log(\lambda/2) + O(\lambda^{-1}) \; . \end{aligned}$$

Improved smaller constants $C_n(\lambda) < C_{n-1}(\lambda)$ with $C_1(\lambda) = C(\lambda)$ could be obtained by estimating

$$n^{-1} \int_{\mathbf{T}^2} \log ||A^n(x,y)|| \; dxdy$$

from above, where n is small. But already for $n = 3$, explicit bounds become more difficult due to the transcendental nature of the integrals. For $n = 2$, one would have to estimate the integral

$$\int_{\mathbf{T}^2} \log || \begin{pmatrix} E - \lambda \cos(y) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E - \lambda \cos(x) & -1 \\ 1 & 0 \end{pmatrix} || \; dxdy$$

from above. With a numerical evaluation of this integral, we get an upper bound for the Lyapunov exponent. The values of $C_n(\lambda)$ could be estimated computer assisted. We expect that like this, one should reach λ_n close to 2 for large n. Maybe one could even get $\lambda_n \to 0$ for $n \to \infty$, if also the lower bound of the determinant $\int_{|w|=1} \log \det L_T K_{S,w}$ could be improved further with the Jensen formula.

7 Uniform hyperbolicity: a result of Ruelle

In this section we look at a class of uniformly hyperbolic cocycles and apply a result of Ruelle [124] to conclude that the lower bound on the Lyapunov exponent depends in our case smoothly on parameters. Furthermore, the result implies that the neighborhood around such a cocycle, for which one has positive Lyapunov exponents, becomes large if the Lyapunov exponent is large.

Consider a $(d+1) \times (d+1)$- matrix-valued map

$$(x,y) \mapsto A(x,y) = \begin{pmatrix} \lambda/2 + a(x,y) & b(x,y) \\ c(x,y) & d(x,y) \end{pmatrix} \; ,$$

where $a(x,y) \in \mathbf{C}$, $d(x,y)$ is a $d \times d$ matrix and $b(x,y)$ and $c(x,y)^T$ are vectors in $\mathbf{C}^d$. All entries are assumed to be bounded and measurable functions. Together with $T \in \mathcal{Y}$, it defines a cocycle $(x,y) \mapsto A^n(x,y)$. We look at cocycles in a neighborhood of $\begin{pmatrix} \lambda/2 & 0 \\ 0 & 0 \end{pmatrix}$, which has one Lyapunov exponent $\log(\lambda/2)$ and $(d-1)$ Lyapunov

exponents $-\infty$.

The maximal Lyapunov exponent in such a uniformly hyperbolic situation (which generalizes the logarithm of the Perron-Frobenius eigenvalue of a positive matrix) depends smoothly on parameters [124]. We formulate this in a special case:

Lemma 7.1 (Ruelle) *The maximal Lyapunov exponent $A \mapsto \mu(A)$ of is Fréchet differentiable and realanalytic in a neighborhood of $A_0 = \begin{pmatrix} \lambda/2 & 0 \\ 0 & 0 \end{pmatrix} \in L^\infty(\mathbf{T}^2, M(d+1, \mathbf{C}))$, where $M(d+1, \mathbf{C})$ is the vector space of all complex $(d+1) \times (d+1)$ matrices. If $w \mapsto A_w$ be a complex parameterization of the cocycle such that for all $\{|w| \leq r\}$, the inequalities*

$$|a_w(x,y)|, ||b_w(x,y)||, ||c_w(x,y)||, ||d_w(x,y)|| \leq \epsilon$$

hold. Then, for fixed λ and small enough $|\epsilon|$, or for fixed $\epsilon > 0$ and large enough λ, the Lyapunov exponent $w \in \{|w| \leq r\} \mapsto \mu(A(w))$ is a harmonic function in w.

Proof. If $|a(x,y)|, ||b(x,y)||, ||c(x,y)||, ||d(x,y)|| \leq \epsilon$ and ϵ is small enough, the cocycle maps the constant cone bundle

$$(x,y) \mapsto C(x,y) = C = \{c \cdot (1,\eta) \mid ||\eta|| \leq 1, c \in \mathbf{C}\ \}$$

strictly into itself.

(One could write the complex matrix cocycle as a real matrix cocycle on $\mathbf{R}^{2(d+1)}$ for which the cone

$$C = \{c \cdot (1,\eta) \mid ||\eta|| \leq 1, \eta \in \mathbf{R}^{2d+1}, c \in \mathbf{R}\ \}$$

in $\mathbf{R}^{2d+2}$ is strictly invariant. This is equivalent to the fact that one can find a basis and $\delta > 0$ such that $|[A(x)]_{ij}| \geq \delta$. By ([124] 4.7), one can deal directly with the complex case.)

[124] is written in the case of continuous cone bundles and homeomorphisms, but remark 4.9 in [124] extends this to the measure theoretic case.

From Proposition 4.8 in [124] we obtain especially:

Corollary 7.2 *For small r, the Lyapunov exponent $(w,z) \mapsto \mu(G(w,z))$ is pluriharmonic (harmonic both in w and z) on $\{(w,z) \mid 0 \leq |w| < r, 0 < |z| < r\ \}$.*

Corollary 7.3 *Given $R > 0$ and $\epsilon > 0$. Let $B_R(A)$ be the ball of radius R around A. For large enough λ, the ball $B_R(G(w,z))$ is contained in the open subset of uniformly hyperbolic cocycles and $|\mu(B) - \mu(A)| \leq \epsilon$ for all $B \in B_R(A)$.*

Proof. The Fréchet derivatives of the function $A \mapsto \mu(A)$ can be computed (see formula (3.6) in [124]) and satisfy $D^n_A \mu(A) \leq C^n/\lambda^n$, where C is a constant independent of λ.

Actually, one only needs the first derivative: Bloch's theorem which says that if a function f is analytic in a disc B_R of radius R, then $f(B_R)$ contains a disc of radius $|Df(a)|Rl$, where l is a constant not depending on anything (see [3] p. 39).

8 Determinants of random difference operators

In this section we look at determinants for classes of random operators. We will see that for a class of random operators, which do not need to be invertible, the determinant is nonnegative and satisfies $\det(LK) = \det(L)\det(K)$. This product formula of Fuglede and Kadison [38] for operators in a type II_1 von Neumann algebra usually assumes the operators L and K to be invertible. In the case of random, finite order difference operators, we can approximate the operators by periodic operators, which have the spectrum on a union of one-dimensional curves. By changing the energy we can approximate the operators by invertible operators for

which the formula holds. The formula holds then for the random difference operators also.

A measure dk in the complex plane is called the density of states of a (not necessarily selfadjoint) random bounded operator $(x,y) \in \mathbf{T}^2 \mapsto L(x,y) = \sum_{j=-p}^{p} a^{(j)}(x,y)\tau^j$ on $l^2(\mathbf{Z}, \mathbf{C}^N)$ with $N \times N$ matrices $a^{(j)}$ and $\tau u_n = u_{n+1}$, if for every polynomial f, one has

$$\int_{\mathbf{T}^2} \mathrm{Trace}([f(L(x,y))]_{00}) \; dx \; dy = \int_{\mathbf{C}} f(E) \; dk(E)$$

where Trace is the usual trace for $N \times N$ matrices normalized in such a way that $\mathrm{Trace}(1) = 1$.

For operators L with a density of states dk one has a trace

$$\mathrm{tr}(L) = \int E \; dk(E) = \int \mathrm{Trace}[L(x,y)]_{00} \; dx \; dy \; .$$

If the Thouless formula holds and the Lyapunov exponents of the transfer cocycles are all finite, then the measure dk has finite logarithmic energy and the potential $E \mapsto \mathrm{tr}(\log|L-E|) = \int_{\mathbf{C}} \log|E-E'| \; dk(E')$ is defined and finite for all E.

In general, if L has a density of states dk, a determinant $\det(L)$ of L can be defined by

$$\log\det(L) = \mathrm{tr}(\log(|L|)) \; .$$

If the value $\int_{\mathbf{C}} \log|E| \; dk(E)$ of dk at the point $E = 0$ should be $-\infty$, one would put $\det|L| = 0$.

(The extension of the determinant to noninvertible operators, which we look at, is what Fuglede and Kadison call an 'analytic extension of the determinant' to singular operators. This is in contrast to the 'algebraic extension' for which the determinant is put always to zero if the operator is noninvertible).

Given $T, S \in \mathcal{Y}$ and $a^{(j)}, b^{(j)} \in L^\infty(\mathbf{T}^2, dxdy)$. Two not necessarily selfadjoint elements $L = \sum_j a^{(j)}(x,y)\tau^j, K = \sum_j b^{(j)}(x,y)\sigma^j$ in the crossed products define random operators

$$(L(x,y)u)_n = \sum_{j=-p}^{p} a_n^{(j)}(x,y)u_{n+j}, \;\; (K(x,y)u)_n = \sum_{j=-p}^{p} b_n^{(j)}(x,y)u_{n+j}$$

with $a_n^{(j)}(x,y) = a^{(j)}(T^n(x,y)), b_n^{(j)}(x,y) = b^{(j)}(S^n(x,y))$.

Lemma 8.1 *The operators L, K, LK have all density of states. The Thouless formula holds for L, K and LK and* $\det(LK) = \det(L)\det(K)$.

Proof. (i) The operators L, K can be embedded in a single von Neumann algebra without changing the determinant.
Proof. L is in the crossed product $\mathcal{Z}_T$ of $L^\infty(\mathbf{T}^2, dxdy)$ with the $\mathbf{Z}$-action $\alpha_T : f \mapsto f(T)$ and K is in the crossed product $\mathcal{Z}_S$ of $L^\infty(\mathbf{T}^2, dxdy)$ constructed with the $\mathbf{Z}$-action $\alpha_S : f \mapsto f(S)$. The operator LK is in the crossed product $\mathcal{Z}_{T\times S}$ of $L^\infty(\mathbf{T}^2 \times \mathbf{T}^2, dxdy \times dxdy)$, constructed with the $\mathbf{Z}$-action $\alpha_{T\times S}$, where $T \times S(x_1, y_1, x_2, y_2) = (T(x_1, y_1), S(x_2, y_2))$ on $\mathbf{T}^2 \times \mathbf{T}^2$. One can embed the two crossed products $\mathcal{Z}_T$ and $\mathcal{Z}_S$ in $\mathcal{Z}_{T\times S}$. The density of states of LK exists in the same way as it exists for L and K. Moreover, the density of states of L (rsp. K) in $\mathcal{Z}_{T\times S}$ is the same as the density of states of L in $\mathcal{Z}_T$ (rsp. in $\mathcal{Z}_S$).

(ii) For the verification of the Thouless formula, see Section 15).

(iii) We can assume without loss of generality that L and K are invertible.
Proof. By ergodic decomposition (i.e. [26]), one can assume T, S to be ergodic.
Approximate $T, S \in \mathcal{Y}$ by periodic transformations T_k using Rholin's lemma [53]: the measure of $\{(x,y) \in \mathbf{T}^2, \; T(x,y) \neq T_k(x,y)\}$ is then smaller than $1/k$ and $T_k^k = Id$. By ergodic decomposition, the determinant is in such a case an integral of determinants of type II_1 factors, in which case L_k, K_k as well as the product

$L_k K_k$ are periodic operators which have their spectrum contained in a finite union of real smooth curves in the complex plane. In this case, there are arbitrarily small complex numbers E_k such that $L_k - E_k, K_k - E_k$ are invertible allowing to apply the determinant formula. Now, $E \mapsto \det(L_k - E), E \mapsto \det(K_k - E)$ as well as $E \mapsto \det((L_k - E)(K_k - E))$ are continuous. $L_k - E_k, K_k - E_k$ are invertible and $\det(L_k - E_k) \to \det(L)$, $\det(K_k - E_k) \to \det(K)$, and $\det((L_k - E_k)(K_k - E_k)) \to \det((L-E)(K-E))$ for $n \to \infty$.

(iv) The product formula.
Proof. We can now invoke the determinant theory for finite type von Neumann algebras [38] (see [27] chapter I) which tells that $\det(LK) = \det(L)\det(K)$ for invertible operators L, K in a type II_1 von Neumann algebra.

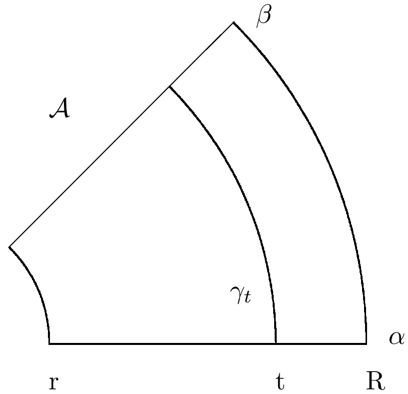
Corollary 8.2 *If $L^{(i)} = a^{(i)}\tau + (c^{(i)}\tau)^* + b^{(i)}$ are random Jacobi matrices over dynamical systems $T^{(i)} : \mathbf{T}^2 \mapsto \mathbf{T}^2$, then*
$$\det(L^{(1)}L^{(2)}) = \det(L^{(1)})\det(L^{(2)}) .$$

Proof. Apply Lemma (8.1).

Whatever Lyapunov exponent estimates for an analytic cocycle can be proven for irrational rotations $x \mapsto x + \alpha$ as in [56, 133], they can up to some correction $C(\lambda)$ also be proven for this cocycle, where the underlying system is an arbitrary measure preserving transformation $T \in \mathcal{Y}$. For example, for any realanalytic nonconstant f, the Lyapunov exponent of $A_{T,\lambda f}$ should be positive for large enough λ.

9 The Jensen formula in a sector

In this section we prove an integrated form of the Cauchy-Riemann differential equations. Unlike the classical Jensen formula which holds in an annulus (for a short proof see [43]), we formulate it in a sector. This allows to apply it in piecewise analytic situations. While we don't use this formula explicitly, we used it to explain some heuristics in the introduction. We furthermore think it could be useful for further improvements on the Lyapunov estimates.



Consider a sector $\mathcal{A} = \{z \in \mathbf{C} \mid |z| \in [r, R], \arg(z) \in [\alpha, \beta]\ \}$ and denote by $\gamma_t = \{|z| = t\} \cap \mathcal{A}$ a circular arc in $\mathcal{A}$. Let g be a bounded continuous, complex-valued function on $\mathcal{A}$ which is analytic and nonzero on $\mathcal{A} \setminus \mathcal{S}$, where $\mathcal{S}$ is a finite set of points in $\mathcal{A}$. Define for $t \in [r, R]$
$$\mathrm{Arg}_{\gamma_t}(g) = \mathrm{Re}\frac{1}{2\pi i}\int_{\gamma_t} \frac{g'}{g}(z)\, dz .$$
(It is defined except possibly at finitely many t, where $\gamma_t \cap \mathcal{S}$ is nonempty.)

Lemma 9.1 (Jensen formula in a sector) *If g is analytic in $\mathcal{A}$ and nonzero in $\mathcal{A} \setminus \mathcal{S}$, where $\mathcal{S}$ is a finite set of points, then*
$$\int_\alpha^\beta \log|g(Re^{2\pi i x})|\, dx = \int_\alpha^\beta \log|g(re^{2\pi i x})|\, dx + \int_r^R \frac{dt}{t}\mathrm{Arg}_{\gamma_t}(g)\, dt .$$

Proof. (i) Assume first that g is analytic in $\mathcal{A}$ and that it has no roots in $\mathcal{A}$. Clearly,
$$\int_\alpha^\beta \log|g(Re^{2\pi i x})|\, dx = \int_\alpha^\beta \log|g(re^{2\pi i x})|\, dx + \int_{\mathcal{A}} \frac{\partial}{\partial t}\log|g(te^{2\pi i x})|\, dt\, dx .$$
Because g is analytic, the Cauchy-Riemann equations $\frac{\partial}{\partial \overline{z}} g(z) = 0$ hold. With $z = t\exp(2\pi i x), g'(z) = (\partial/\partial z)g(z)$, we obtain
$$\frac{\partial}{\partial t}\log|g(te^{2\pi i x})| = \mathrm{Re}\left(\frac{g'(te^{2\pi i x})}{g(te^{2\pi i x})}e^{2\pi i x}\right) = \mathrm{Re}\left(\frac{g'(z)}{g(z)}\,\frac{z}{t}\right) .$$

We have therefore, using $2\pi i dx = dz/z$,

$$\begin{aligned}
\int_\alpha^\beta \log|g(Re^{2\pi ix})|\,dx - \int_\alpha^\beta \log|g(re^{2\pi ix})|\,dx &= \mathrm{Re}\frac{1}{2\pi i}\int_r^R \int_{\gamma_t} \frac{g'(z)}{g(z)}\,\frac{dz}{t}\,dt \\
&= \int_r^R \mathrm{Arg}_{\gamma_t}(g)\,\frac{dt}{t}\ .
\end{aligned}$$

(ii) In general, we partition $\mathcal{A} = [r,R] \times [\alpha,\beta]$ into n^2 small sectors $\mathcal{A}_{jk} = [r_j, r_{j+1}] \times [\alpha_k, \alpha_{k+1}]$ and apply the formula to small sectors $\mathcal{A}_{jk}$ which do not intersect S. The assumption on $\mathcal{S}$ and g assures that for all the integrals, the contribution of the sectors which intersect $\mathcal{S}$ can be neglected in the limit $n \to \infty$.

Remarks.
1) The Jensen formula in the annulus [133, 10] follows: if $\mathcal{A}$ is the annulus and if g is analytic in $\{\arg(z) \in (\alpha_i, \alpha_{i+1}),\ |z| \in (r,R)\ \}$ and if $g(z) = \prod_{j=1}^m (z-a_j)h(z)$ such that $h(z) \neq 0$ everywhere on $\mathcal{A}$, then

$$\int_0^{2\pi} \log|g(Re^{2\pi ix})|\,dx - \int_0^{2\pi} \log|g(re^{2\pi ix})|\,dx = \int_r^R \frac{dt}{t}\mathrm{Arg}_{\gamma_t}(h)\,dt + \sum_{j=1}^m \log(\frac{R}{|a_j|})\ .$$

2) The Jensen formula for subharmonic functions in the annulus $\mathcal{A}$ follows: for subharmonic functions $z \mapsto \log|g(z)| = \int \log|z-z'|\,dk(z')$ in the disc $\{|z| < R\}$, one has

$$\int_0^{2\pi} \log|g(Re^{2\pi ix})|\,dx = \int_0^{2\pi} \log|g(re^{2\pi ix})|\,dx + \int_{\mathcal{A}_{r,R}} \log(\frac{R}{|z|})\,dk(z')\ .$$

3) Lemma (9.1) holds for sets S which are bigger than just a finite point set. For example, S can be a subset of a finite union of smooth curves.

10 Toral homeomorphisms and Lax approximation

In this section we review an approximation result for homeomorphisms on the torus. It allows to approximate the group of measure preserving homeomorphisms by a finite group of piecewise linear transformations. Investigating the Lyapunov map $T \mapsto \mu(A_{T,\lambda\sin})$ on finite groups $\mathcal{Y}_n$ of measure preserving transformations was important for us because the evaluation of the Lyapunov exponent is a finite dimensional integration and quite reliable.

On the group $\mathcal{Y}$ of all measurable, invertible transformations on the torus $\mathbf{T}^d$ which preserve the Lebesgue measure, one has the metric

$$\rho(T_1, T_2) = |\mathrm{dist}(T_1(x), T_2(x))|_\infty\ ,$$

where dist is the geodesic distance on the flat torus and where $|\cdot|_\infty$ is the L^∞-norm.

A cube exchange transformation on $\mathbf{T}^d$ is a periodic, piecewise affine measure-preserving transformation which permutes rigidly all the cubes $\prod_{i=1}^d [k_i/n, (k_i+1)/n]$, where $k_i \in \{0, \dots, n-1\}$. Every point in $\mathbf{T}^d$ is periodic. Such a transformation is determined by a permutation of the set $\{1, \dots, n\}^d$. If the permutation is cyclic, the exchange transformation is called cyclic. A theorem of Lax [82] states that every $T \in \mathcal{X}$ can be approximated in the metric ρ by cube exchange transformations. Actually, T_k can be chosen to be cyclic [2].

Lemma 10.1 (Lax approximation) *For every $\epsilon > 0$ and every homeomorphism $T \in \mathcal{X}$, there exists a cyclic cube exchange transformation T_k which satisfies $\rho(T_k, T) \leq \epsilon$.*

Lax's 'book proof' of this result uses Hall's marriage theorem in graph theory (for a 'book proof' of the later theorem, see [1]).

Remarks.
1) Periodic approximations of symplectic maps work surprisingly well for relatively small n (see [120]). On

the Pesin region this can be explained in part by the shadowing property [65]. The approximation by cyclic transformations make however long time stability questions look different [51].

2) It has been measured that certain Lax approximations T_n of the Standard map have longest orbits of size $\geq \delta n^2$ with $\delta > 0$ [158].

3) If $T \in \mathcal{Y}_k$, the density of states $dk(x,y)$ of a periodic operator $L(x,y)$ is contained in a finite union of curves. The density of states of L is the measure $\int_0^{1/k} \int_0^{1/k} dk(x,y)\; dxdy$ which has positive area in the complex plane. For a cyclic T_k, the Lyapunov exponent is computed as

$$\mu(A_{T_k}) = k^{-2} k^2 \int_0^{1/k} \int_0^{1/k} \log \lambda_{\max}(A^{k^2}(x,y))\; dxdy \; ,$$

where $\lambda_{max}(A)$ is the maximal eigenvalue of a matrix A. The factor k^2 comes because we integrate over a region with Lebesgue measure k^{-2}.

11 Thouless formula for nonselfadjoint operators

In this section we look at the Thouless formula for random operators which can have spectrum with positive area in the complex plane. Nonselfadjoint operators appear naturally, when a problem is parameterized by an external complex parameter like for operators (1). The proof of the Thouless formula which we are looking at in this section relies on the fact that for a general parameterization, there is an abstract Thouless formula which relies on the Riesz decomposition theorem in potential theory.

Consider an analytically parameterized, not necessarily selfadjoint, random Jacobi matrix

$$(L_z(x)u)_n = u_{n+1} + u_{n-1} + f_z(x_n)u_n \tag{6}$$

with a complex potential f_z parameterized by a complex parameter z. Examples are the w- parameterization

$$f_w(x) = (w^{-1}\exp(ix) + w\exp(-ix))/2 \tag{7}$$

for $\mathbf{C} \setminus \{0\}$ and for fixed $w \neq 0$, the energy parameterization

$$E \mapsto f_{E,w}(w,x) = f_w(x) - E \; . \tag{8}$$

Proposition 11.1 (Thouless formula for nonselfadjoint operators) *In the energy parameterization case (8), the Riesz measure of the Lyapunov exponent $\mu(A_{T,E})$ of the transfer cocycle $A_{T,E}$ is the density of states of the operator (1).*

Proof. For any general analytic parameterization $z \mapsto L(z)$, the Lyapunov exponent $\mu(A_z)$ of $L(z)$ is a a subharmonic function of z. The Riesz decomposition theorem (see i.e. [54, 121]) implies the abstract Thouless formula $\mu(z) = \int \log|z - z'|\; dk(z') + h(z)$, where $z \mapsto h(z)$ is harmonic.

The name 'abstract Thouless formula' is from [56].

If $z \mapsto \mu(z)$ is harmonic for large $|z|$ and grows like $\log|az|$ for $|z| \mapsto \infty$ then $\mu(z) = \log(a) + \int \log|z - z'|\; dk(z')$.

The Riesz measure dk of the subharmonic function $z \mapsto \mu(z)$ satisfies $dk = \Delta\mu$ in the sense of distributions, where Δ is the Laplacian on distributions. The measure dk coincides in the periodic case with the density of states because in this case the Lyapunov exponent is zero exactly on the spectrum which implies that the density of states is the equilibrium measure on the spectrum.

Consider the Riesz measures $dk_n(z) = \Delta\mu_n(z)$ of the subharmonic functions

$$z \mapsto \mu_n(z) = n^{-1} \int_{\mathbf{T}^2} \log|(A^n_{z,T}(x,y))_{11}|\; dx\; dy$$

and let $dk(z) = \Delta\mu(z)$ be the Riesz measure of the Lyapunov exponent $z \mapsto \mu(A_{z,T})$. Because $\mu_n(A_{z,T})(x,y)$ converges by the multiplicative ergodic theorem to $\mu(A_{z,T})(x,y)$ for almost all (x,y), the convergence holds also in the sense of distributions. Then also the Riesz measures $dk_n = \Delta\mu_n$ converge to $\Delta\mu$ as distributions for $n \to \infty$. But then, because smooth functions are dense in all continuous functions, $dk_n = \Delta\mu_n$ converges weakly as measures to $dk = \Delta\mu$.

In the energy parameterization case (8), the density of states of L is defined as the measure dk in the parameter plane $\mathbf{C}$ satisfying $\int_{\mathbf{T}^2} [g(L(x,y))]_{00} \; dxdy = \int g(E') \; dk(E')$ for every continuous function g.

For E outside the spectrum of L, the sequence $\int_{\mathbf{C}} (E'-E)^{-1} \; dk_n(x,y,E')$ converges for $n \to \infty$ uniformly to $\int_{\mathbf{C}} (E'-E)^{-1} \; dk(x,y,E')$. Because

$$\mathrm{tr}((L^{(n)}(x,y) - E)^{-1}) = \int_{\mathbf{C}} (E'-E)^{-1} \; dk_n(x,y,E') \; ,$$

where the Riesz measure $dk_n(x,y) = \Delta\mu_n(x,y)$ is a finite point measure located on the point spectrum of $L^{(n)}(x,y)$ defined in an earlier section, the formula holds also after integration over $\mathbf{T}^2$ and dk_n is the density of states of the random operator $(x,y) \mapsto L^{(n)}(x,y)$.

The functions $g_j(E) = (E - z_j)^{-1}$, where $\{z_j\}_{j \in \mathbf{N}}$ is a dense set in the complement of the spectrum of L, span a dense set in $C(\mathbf{C})$. For such functions, we have

$$\int_{\mathbf{T}^2} [g_j(L^{(n)}(x,y))]_{00} \; dxdy \mapsto \int_{\mathbf{T}^2} [g_j(L(x,y))]_{00} \; dxdy$$

by the Avron-Simon lemma.

We know therefore $\int_{\mathbf{T}^2} [g(L(x,y))]_{00} \; dxdy = \int g(E') \; dk(E')$ for all $g \in C(\mathbf{C})$ so that dk is the density of states.

Examples.
1) If $w \mapsto B(w) = w^l A(w)$ is analytic in $\mathbf{C}$, where $A(w)$ is the transfer cocycle of L_w, then the Lyapunov exponent of B satisfies

$$\mu(B(w)) = C + \int_{\mathbf{C}} \log|w - w'| \; dk(w')$$

where dk has total mass $(l+1)$ and where C is a constant.
2) For periodic operators, where the support of the Riesz measure is on spectral bands, the Lyapunov exponent of $w \mapsto A(w)$ is $\mu(w) = \mathrm{Re}[i \arccos(\mathrm{Tr}(A^n)(w)/2)]$. With the parameterization (7), we have

$$\mu(A(w)) = -\log|w| + \int_{\mathbf{C}} \log|w - w'| \; dk(w') + \log|\lambda/2|$$

because $w \mapsto \mu(A(w))$ grows like $\log|w\lambda/2|$ for $|w| \to \infty$ and $dk(\mathbf{C}) = 2$. The Lyapunov exponent of $B(w) = wA(w)$ which is harmonic near 0 satisfies then

$$\mu(B(w)) = \int_{\mathbf{C}} \log|w - w'| \; dk(w') + \log|\lambda/2| \; . \tag{9}$$

We expect the function $w \mapsto \mu(B(w))$, which is harmonic near $w = 0$ to extend often to a harmonic function in the entire complex plane.

Remark. By the Hadamard three circle theorem [10], the function $\mu_{max}(r) = \max_{|w|=r} |\mu(w)|$ is a increasing function in r which is convex in $\log(r)$. Aiming to get rid of the $C(E,\lambda)$ term in Conjecture (1.1), it would be good to know $M = \{|w| = 1 \mid \mu(w) = \mu_{max}(|w|) \}$ which is contained in the subset of the unit circle on which the potential is $\geq u(0)$. In the case of the Standard map, a candidate for a maximum on $|w| = 1$ is $w = 1$ because the pointwise Lyapunov exponent $\mu(x,y,w)$ satisfies $\mu(x,y,w) = \mu(-x,-y,\overline{w})$ so that $\mu(w) = \mu(\overline{w})$. See 18) in the discussion section.

12 Log-Holder continuity and capacity

In this section we consider a potential theoretical property of the Riesz measure of the Lyapunov exponent. This applies to the Riesz measure of the parameterization $w \mapsto L_w$ or, for fixed w to the energy parmeterisation $E \mapsto L_w - E$, where the Riesz measure dk is the density of states.

As in the selfadjoint case, the fact that the Lyapunov exponent is nonnegative implies mild regularity for the measure dk. Log-Holder continuity is a property of measures with finite potential theoretical energy. It implies that such a measure is absolutely continuous with respect to the zero-dimensional Hausdorff measure [35].

Lemma 12.1 (Log-Holder continuity) *Let $w \mapsto L_w$ be an analytic parameterization of a random Jacobi operator. The Riesz measure dk of the nonnegative subharmonic function $w \mapsto \mu(A_T(w))$ is log-Holder continuous: there exists a constant C such that*

$$dk(B) \leq C \cdot (\log(1/|B|))^{-1}$$

for every ball B with diameter $|B|$. The constant C only depends on L and $|E|$.

Proof. The proof is the same as in the real case [22, 17]: assume dk is contained in a ball of radius c and assume that B is any ball of radius $r < 1$ around $z \in \mathbf{C}$. With $A = \{z' \mid |z' - z| \geq 1\}$, we have

$$0 \leq \int_A \log|z - z'| \, dk(z') \leq \log(1 + c + |z|) \; .$$

Therefore, using in the first inequality of the following identities also that $|z - z'| \leq 1$ on A^c and $\int_{\mathbf{C}} \log|z - z'| \, dk(z') \geq 0$, we have

$$0 \geq \int_{A^c} \log|z - z'| \, dk(z') = \int_{\mathbf{C}} \log|z - z'| dk(z') - \int_A \log|z - z'| \, dk(z') \geq 0 - \log(1 + c + |z|) \; .$$

For $z_1 \neq z$ with $|z_1 - z| = r$, and because $|z - z'| < |z - z_1|$ for $z' \in A^c$

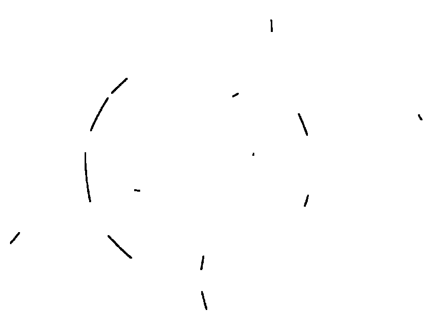
$$\log|z - z_1| \, dk(B) \geq \int_{A^c} \log|z - z'| \, dk(z') \geq -\log(1 + c + |z|)$$

so that $0 \leq \log(1 + c|z|) + \log|z - z_1| \, dk(B)$ which is the claim with $C = \log(1 + c|z|)$.

Example. Consider the case, when $T = T_k$ is a Lax cube exchange transformation on the torus and where the operator

$$L_w(x, y) = \tau + \tau^* + \lambda/2(w^{-1} z_j + w z_j^{-1})$$

with $z_j = z_j(x, y) = \exp(i x_j), (x_j, y_j) = T_k^j(x, y)$ is a periodic, nonselfadjoint Jacobi matrix.



We are interested in the potential theory of the w-spectrum

$$\sigma(L(x, y)) = \{w \mid \mathrm{tr}(A_w^n(x, y)) \in [-2, 2]\} \; ,$$

which supports the 'w-density of states' $dk(x, y)$ of $L(x, y)$. In the picture to the left, we see an example of the 'w-spectrum' $\sigma(L(x, y))$ for $\lambda = 2.1$ where $k = 7$. One can see six bands located on the unit circle and eight bands away from the unit circle.

The 'w-density of states' of L is then $dk = \int_{\mathbf{T}^2} dk(x, y)$ and its support, the 'w-spectrum' of the random operator L is the union $\bigcup_{x, y \in \mathbf{T}^2} \sigma(L(x, y))$ and has in general positive area in the complex plane.

The potential theoretical energy

$$I(dk(x,y)) = -\int_{\mathbf{C}}\int_{\mathbf{C}} \log|z-z'|\; dk(x,y)(z')\; dk(x,y)(z)$$

is $I(dk(x,y)) = \log(\lambda/2)\; dk(x,y)(\mathbf{C}) = 2\log(\lambda/2)$ because $dk(x,y)(\mathbf{C}) = 2$ and the Thouless formula assures that the Lyapunov exponent $\mu(x,y)(z)$ is related to the potential $u(x,y)(z) = \int_{\mathbf{C}} \log|z-z'|\; dk(x,y)(z')$ of $dk(x,y)$ by $\mu(x,y)(z) = \log(\lambda/2) + u(x,y)(z) - \log(w)$ and the Lyapunov exponent is zero on the spectrum $\sigma(L(x,y))$. The measure $dk(x,y)/2$ is the equilibrium measure on the set $\sigma(L(x,y))$. It has the potential theoretical energy $\log(\lambda/2)/2$ and the capacity $C(\sigma(L(x,y)))$ is $\sqrt{\frac{2}{\lambda}}$.

13 Harmonic continuation of some singular integrals

In the case of the Standard map, the function $\mu(w) = \log(w\ \det(L_w))$ with L_w given by (1) is for small $|w|$ the real part of an analytic function. By the Cauchy-Riemann differential equations, the $\arg(w)$-dependence of the function $w \mapsto \exp(\mu(w))$ is related to the change of the harmonic conjugate along the radial direction. The Jensen formula extends this to the w-spectrum of L_w, where μ is no more analytic. There are cases (and we expect it to be the case in the Standard map), in which the harmonic conjugate can be extended as a harmonic function on a neighborhood of the unit disc. Estimating this harmonic conjugate with complex analytic methods could lead to possibly better lower bounds of the Lyapunov exponent. Alternatively, one could look at the argument change (which differs from the harmonic conjugation) and use the Jensen formula directly to estimate the oscillation of the Lyapunov exponent $w \mapsto \mu(A(w))$ for w the unit circle. In this section we formulate the problem of computing the harmonic continuation rsp. the argument change along radial lines, when the w-spectrum is located on the unit circle. In the next section, we will see that assuming dk to be supported on $\mathbf{T}$ is no big loss of generality for estimation purposes because projecting the spectrum onto the unit circle does not change much the potential (which is the Lyapunov exponent.)

A continuous, real-valued periodic function $x \mapsto \phi(x)$ on the circle defines a complex-valued function $x \mapsto a(x) = \exp(i(x+\phi(x)))$ and a measure $dk = h(x)dx$ on the circle $\mathbf{T}$ by pushing forward the Lebesgue measure $dk = a^* dx$. Denote by $a_n = \int \exp(-inx)\ h(x)dx$ the Fourier coefficients of this measure. The function $f(z) = \int_{\mathbf{T}} dk(x')/(z - \exp(ix')) = \int_{\mathbf{T}} dx/(z-a(x))$ defines two analytic functions $g_\pm(z)$, where g_+ is analytic on $|z| < 1$ and $g_-(z)$ is analytic in $|z| > 1$. These functions are defined by their Taylor expansions at $z_+ = 0$ and $z_- = \infty$:

$$\begin{aligned} h_+(z) &= -\int_{\mathbf{T}} a^{-1}(x)\frac{1}{1-z/a(x)}\; dx = -\sum_{n=0}^{\infty} z^n a_{-(n+1)}, \\ h_-(z) &= \int_{\mathbf{T}} z^{-1}\frac{1}{1-a(x)/z}\; dx = \sum_{n=0}^{\infty} z^{-(n+1)} a_n = \sum_{n=-1}^{-\infty} z^n a_{-(n+1)} \;. \end{aligned}$$

The function $h = h_- - h_+ = \sum_{n\in\mathbf{Z}} z^n a_{-(n+1)}$ is like this represented by the boundary value of two analytic functions $h_\pm$. One has for example that if h is in L^p, then $h_\pm$ are in the Hardy space H^p. The function h is realanalytic if and only if h is analytic in some strip around the unit circle. In this case, we can write $f(z) = h_+(z) = h(z) + h_-(z)$ for $\{1-\epsilon < |z| \leq 1\}$ and $f(z) = h_-(z) = -h(z) + h_+(z)$ for $\{1 < |z| < 1+\epsilon\}$. The functions $h_\pm$ can then be analytically continued to a larger domain. If dk extends to an entire function, then both $h_\pm$ extend to entire functions.

The integral $H(z) = \int_{\mathbf{T}} \log(z - a(x))\; dx$ with $H(0) = 0$ can be expanded in the same way at $z = 0$ and ∞ and

$$H = H_- - H_+ = \sum_{n\in\mathbf{Z}} z^{n+1} a_{-(n+1)}/(n+1) = \sum_{n\in\mathbf{Z}} z^n a_{-n}/n \;.$$

The imaginary part of H on $|z| = 1$ is the average argument change from 0 to ∞ of the function $a \mapsto z - a(x)$ along the line going through $z = \exp(i\alpha)$.

Near $z_- := 0$, the function $\mathrm{Im}(H_-(z))$ is the argument change of $\exp(H_-(z))$ from 0 to z. Near $z_+^{-1} := 0$, the function $\mathrm{Im}(H_+(z))$ is the argument change of $\exp(H_+(z))$ from ∞ to z.

Because of the Jensen formula (9.1), we are interested in the argument change $\alpha(z)$ from 0 to z along a straight line of the function $\int_{\mathbf{T}} \log(z - a(x))\, dx$. For each point $a(x)$, the argument change $\alpha(z)$ from 0 to z is a well defined angle. For $|z| < 1$, we have $\alpha(z) = \mathrm{Im}(H_-(z))$. For $|z| > 1$ we can write $\alpha(z) = \mathrm{Im}(H_+(z)) + \beta(z)$, where $\beta(z) = \beta(z/|z|)$ is the argument change along the straight line from 0 to ∞ which passes through z. Now $\beta(z) = \mathrm{Im}(H(z/|z|))$ so that

$$\begin{aligned} \alpha(z) &= \mathrm{Im}(H_+(z)) + H(z/|z|))\ ,\ |z| \geq 1 \\ \alpha(z) &= \mathrm{Im}(H_-(z))\ ,\ |z| \leq 1\ . \end{aligned}$$

14 Lipshitz approximation of the Lyapunov exponent

In the theory of Schrödinger operators one uses the fact that the Lyapunov exponent $\mu(A_E)$ of the transfer cocycle A_E of L is the Hilbert transform of the integrated density of states (or rotation number [123, 25]) $\rho(E) = dk(-\infty, E)$, where dk is the density of states of L. For the spectral problem $w \mapsto L_w$, where the w-spectrum is contained in an annulus $\mathcal{A}_r = \{|z| \in (r, r^{-1})\ \}, r < 1$ containing the unit circle, one can ask for the regularity of the map $\alpha \mapsto \mu(L(\exp(i\alpha))$. The situation for the unit circle in the last section is relevant to this problem. The idea is to project the measure dk onto the unit circle with the map $\pi(z) = \arg(z)$. While this does not change the Lyapunov exponent much, one can use the classical properties for the new Lyapunov exponent which is the potential of a measure on the unit circle. A Lipshitz continuity of the 'integrated density of states' $\alpha \mapsto dk(\{z \mid \arg(z) \in (0, \alpha)\})$ would imply that $\alpha \mapsto \mu(\alpha) = \mu(L(\exp(i\alpha)))$ would be close to a Lipshitz continuous map.

Let dk be the Riesz measure of the Lyapunov exponent $\mu(w)$ of $w \mapsto L_w$ so that $\mu(w) = \int_{\mathbf{C}} \log|z - z'|\, dk(z')$. Let $\pi^* dk$ be the measure on the unit circle obtained as the push-forward of dk under the projection $\pi(r \exp(i\phi)) = \exp(i\phi)$.

The fact that the Hilbert transform preserves Lipshitz continuity can be derived from the Jensen formula in a sector (9.1) (rsp. the Cauchy-Riemann differential equations). Let H be the function as in the last section with the measure $\pi^* dk$ on the unit circle. The statement is that if the radial argument change $\mathrm{Im}(H(z))$ of the function $z \mapsto g(z) = \exp(H(z))$ is Lipshitz continuous on the unit circle and the Lipshitz constant is C, then the function $\mathrm{Re}(H(z)) = \int \log|z - z'|\ \pi^* dk(z')$ is Lipshitz continuous on the unit circle with Lipshitz constant C.

Lemma 14.1 *If dk is supported on $\mathcal{A}_r = \{|z| \in (r, r^{-1})\}$ and dk is the Riesz measure of a Lyapunov exponent, then*

$$\int_{\mathbf{C}} \log|z - z'|\, dk(z') \geq \int_{\mathbf{C}} \log|z - z'|\ \pi^*(dk)(z') - d(r)\ .$$

where $d(r) \to 0$ for $r \to 1$.

Proof. Given a point z_0 with $|z_0| = 1$. The set $Y(z_0, r) = \{w \in \mathcal{A}_r \mid |w - z_0| > |\pi(w) - z_0|\ \}$ is an open set. We have $Y(z_0, r) \subset Y(z_0, r')$ for $r < r'$. Write

$$\int_{\mathbf{C}} \log|z - z'|\, dk(z') = \int_{Y(z_0,r)} \log|z - z'|\, dk(z') + \int_{\mathcal{A}_r - Y(z_0,r)} \log|z - z'|\, dk(z')\ .$$

We have $\int_{\mathcal{A}_r - Y(z_0,r)} \log|z - z'|\, dk(z') \geq \int_{\mathcal{A}_r - Y(z_0,r)} \log|z - z'|\ \pi^* dk(z')$ and

$$\int_{Y(z_0,r)} \log|z - z'|\, dk(z') \geq \int_{Y(z_0,r)} \log|z - z'|\ \pi^* dk(z') - d(r)\ ,$$

where $d(r) \to 0$.

A numerical bound on the Lipshitz constant of $\alpha \mapsto \int_0^\alpha \pi^*(dk)(\beta)\ d\beta = dk(\{z \mid \arg(z) \in (0, \alpha)\})$, where dk is the Riesz measure of the Lyapunov exponent $w \mapsto \mu(A_{E,\lambda f,T}(w))$ would allow to estimate the Lyapunov exponent of the transfer cocycle $A_{E,\lambda f,T}$ of the operator (1).

15 An estimate for harmonic maps

We have numerical evidence that in the case of the Standard map the Lyapunov exponent $\mu(w)$ is the sum of a positive subharmonic function and a harmonic function obtained by harmonic continuation of $\mathrm{Re}\int \log(w - w')\, dk(w') + \log|w|$ from a neighborhood of $\{w = 0\}$. If this should turn out to be true, estimates for harmonic maps could be used to give alternative and maybe better estimates for the Lyapunov exponent.

For a general harmonic function h, one can estimate $\inf_{|z|=r} h(z)$ in terms of $C(r) = \sup_{|z|=r} h(z)$. We present two ways. The first way uses Harnack's inequality. A second approach uses the distortion theorem for univalent functions.

Lemma 15.1 (Lower bound for harmonic maps) *Let h be a harmonic map in a disc $\mathbf{D}(r') = \{|z| < r'\}$ with $r' > r, r' > 1$, such that $h(0) = 0$ and $h(z) < C(r)$ for all $z \in \mathbf{D}(r)$. Then a)*

$$h(z) \geq -C(r)\frac{2}{r-1} \; .$$

for $|z| = 1$ and b)

$$h(z) \geq -C(r)(1 + \frac{(1+r^{-1})}{(1-r^{-1})^3})$$

for $|z| = 1$.

Proof. a) $v(z) = C(r) - h(z)$ is nonnegative in the disc $\mathbf{D}(r) = \{|z| < r\}$ and $v(0) = C(r)$. Harnack's inequality gives

$$C(r) - h(z) = v(z) \leq C(r)\frac{r+1}{r-1}$$

so that $h(z) - C(r) \geq -C(r)\frac{r+1}{r-1}$ and $h(z) \geq C(r)(1 - \frac{r+1}{r-1}) = -C(r)\frac{2}{r-1}$.

b) The function $C(r) - h(z)$ is positive in $\mathbf{D}(r)$. Let $H(z)$ be the analytic function in $\mathbf{D}(r)$ which has h as its real part and which vanishes at 0. By the Noshiro-Warschawski criterium [33], the analytic function

$$F(z) = \int_0^z (C(r) - H(z))\, dz$$

is univalent in $\mathbf{D}(r)$ because it has a derivative which has a positive real part in $\mathbf{D}(r)$. The function $\tilde{F}(z) = F(rz)/(rC(r))$ is in the class S of functions which are analytic and univalent in the unit disc and satisfy $\tilde{F}'(0) = 1$. By the distortion theorem for maps in the class S, we have

$$|\tilde{F}'(z)| \leq \frac{(1+R)}{(1-R)^3}$$

for $|z| = R < 1$. Assuming $r > 1$ and applying this for $R = r^{-1}$, we get for $|z| = R$

$$|\tilde{F}'(z)| = |\frac{F'(rz)}{C(r)}| = |1 - \frac{H(rz)}{C(r)}| \leq \frac{(1+r^{-1})}{(1-r^{-1})^3} \; .$$

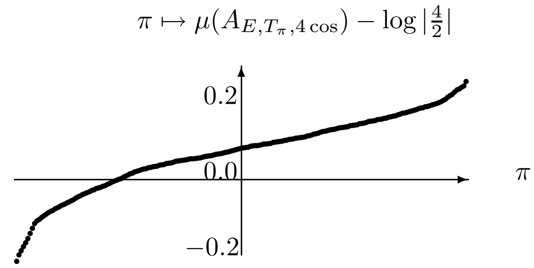
In other words, if $|z| = 1$, then

$$|1 - \frac{H(z)}{C(r)}| \leq \frac{(1+r^{-1})}{(1-r^{-1})^3}$$

which leads to $|H(z)| \leq C(r)(1 + \frac{(1+r^{-1})}{(1-r^{-1})^3})$ and implies the claim. ∎

16 Remarks

1) **Measure preserving maps**.



All cyclic permutations T_π of $\{1, \cdots, 9\}$. They are ordered such that the excess to $\log|\lambda/2|$, shown here for $\lambda = 4$, increases.

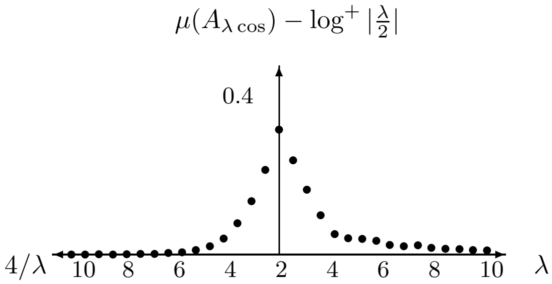
The random variables x_n where $T^n(x, y) = (x_n, y_n)$ have as their law the Lebesgue measure. One could enlarge the class of transformations even so it is not much gain of generality [87]. The conjectured lower bound $\log(\lambda/2)$ can be expected to hold in general only for homeomorphisms $T \in \mathcal{X}$. The reason is that for some piecewise periodic measure preserving transformations $T_\pi : (x, y) \mapsto (x + \pi(x)/n \bmod 1, y)$ with a permutation π on $\{1, 2, \dots, n\}$, the value of $\mu(A_{T,\lambda\cos})$ can be slightly smaller than $\log(\lambda/2)$. The Figure to the left shows the distribution of the random variable $\pi \mapsto \mu(A_{T_\pi,\lambda\cos})$ on the group $\mathcal{Y}_9 \subset \mathcal{Y}$ of all permutations of the 9 annuli $\{(j/9, (j+1)/9] \times \mathbf{T} \subset \mathbf{T}^2\}_{j=1}^{9}$ in the case $\lambda = 4.0$.

It would be interesting to know $\inf_{T\in\mathcal{Y}} \mu(A_{T,\lambda\cos})$ for given $\lambda > 0$ and whether the infimum is attained. By upper continuity of the Lyapunov exponent the infimum taken on the discrete set $\bigcup_n \mathcal{Y}_n \subset \mathcal{Y}$ is the same as the infimum taken on $\mathcal{X}$.

2) **Hamiltonian flows**.
Like for a large class of monotone twist maps, any Standard map is the time-one map of a Hamiltonian differential equation with time-dependent periodic Hamiltonians [106]. This is also true in higher dimensional cases [11, 49]. If positive Kolmogorov-Sinai entropy is stable under perturbations of the map, there would be many real-analytic, time-dependent periodic potentials V for which the time-dependent Hamiltonian system $\ddot{x} = V(t, x)$ has positive Kolmogorov-Sinai entropy. To see this, we note that the potential $V(t, x) = \delta_{t\in\mathbf{Z}}\lambda \sin_m(x)$ (which is a distribution in t) has as a Poincaré map the Standard map $T_{\lambda\sin_m}$. The smoothed-out potentials $V_\epsilon(t, x) = \phi_\epsilon(t) * V(t, x)$, where ϕ_ϵ is a mollifier function, has a Poincaré map which is realanalytic in (x, y) and which is arbitrarily close to $T_{\lambda\sin_m}$ in a Banach space of realanalytic maps. For fixed λ, the Poincaré map has now positive metric entropy for small ϵ and large m because the stability result as stated in Conjecture (1.1) extends to general realanalytic perturbations of the map (and not only to perturbations of the function f). It would be nice to see entropy estimates for Hamiltonian flows on three dimensional energy surfaces of systems $\ddot{x} = V(x)$, where $V(x)$ is a potential on $\mathbf{T}^2$ or to geodesic flows on the torus $\mathbf{T}^2$ or to real-analytic, strictly convex Birkhoff billiards for which we have no example, for which positive Kolmogorov-Sinai entropy has been proven.
Even more off-limit seem problems in celestial mechanics like the Störmer problem of particles trapped in the van Allen belts of the Earth's magnetic dipole field [14] (for which Pesin theory in the form [66] can be shown to apply) or particular Newtonian three body problems.

3) **Smaller** λ.
Estimating the entropy in the case of small λ is also open for the Chirikov Standard map. Experiments indicate that $\mu(\lambda) > 0$ for all $\lambda > 0$.

Numerically, we find $\mu(\lambda) \geq \log(\lambda/2)$ for all λ and as a "rule of thumb" that the averaged Lyapunov exponent $\mu(\lambda)$ satisfies almost but not exactly the Aubry duality $\mu(\lambda) \sim \log(\lambda/2) + \mu(4/\lambda)$.



How close are we to Aubry duality?
For each λ we averaged 70x70=4900 different orbits of length 10^6 on the phase space.

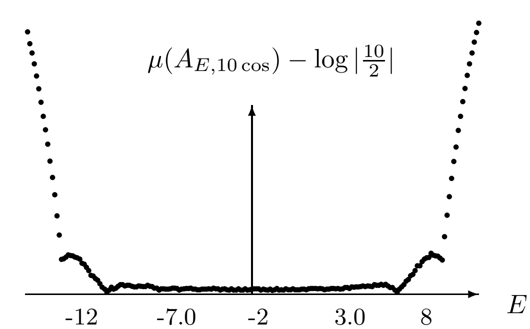
The duality holds in the case of the dynamical system T_0 (the almost Mathieu case). It does not hold for $T_{\lambda \sin}$ as the Figure to the left shows for $\lambda \in [4/10, 2]$ and $\lambda \in [2, 10]$. Indeed, by computing moments of the density of states of L using a random walk expansion [73], one can check that the discrete random Schrödinger operator $L_{T,\lambda\cos} = \Delta + \lambda\cos(x)$ over the dynamical system T_λ does not have the same density of states as the naive "dual operator" $\tilde{L} = (\lambda/2)\Delta + 2\cos(x) = (\lambda/2)L_{T,(4/\lambda)\cos}$. (It is not excluded however that changing T in $\tilde{L}$ could restore a generalized duality.)

As in the almost Mathieu case, there is a two-dimensional magnetic operator involved. The magnetic field at a plaquette $P_{n,m}$ is $\exp(i(x_n - x_{n-1})$ independent of m. In the Standard map case, the magnetic fields depend on space and are correlated. See 25) for more on the Aubry duality.

4) **Smooth cocycles**.
Results like in [154, 156] depend on the dynamical system, e.g. Diophantine conditions for irrational rotations. It is known that positive Lyapunov exponents of $SL(2,\mathbf{R})$ cocycles can be destroyed away from Anosov systems by measurable perturbations of the cocycle [68], by continuous perturbations [39] or using the unproven "Last theorem of Mané" [95], by C^1-perturbations of a non-Anosov map.

5) **Equilibrium measures**.
Via the variational principle (see e.g. [143]) metric entropy results give lower bounds on the topological entrop. with previous known lower bounds [32, 75]. The estimates in [75] are obtained using a Gronwall argument which estimates the spectral radius of the operator L^{-1} (where L is defined by a hyperbolic invariant measure of the Chirikov map) and which are far from optimal (the bounds in [75] are less good than the estimates in [32] but are proven when f is a nonconstant Morse function and not only in the case of the Standard map where $f = \sin$.) The equilibrium measure is a T-invariant measure on $\mathbf{T}^2$ which maximizes the entropy and has as the metric entropy the topological entropy. In the case of the Standard map, it is not known whether there is an absolutely continuous equilibrium measure.

6) **Absolutely continuous spectrum**.



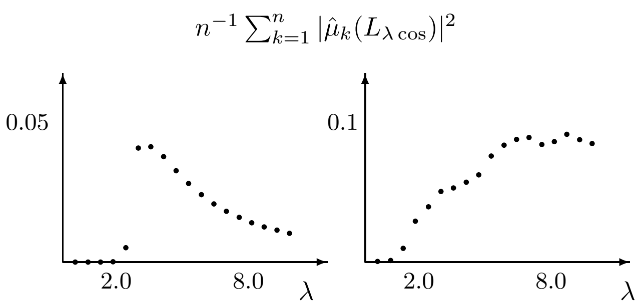
The Lyapunov exponent of the Chirikov Hamiltonian.
The graph shows a nonnegative excess to $\log|\lambda/2|$ for all E.

Lyapunov exaponents have applications in the theory of discrete random Schrödinger operators [23, 17, 116]. The attribute 'random' is used in that theory often in the same way as 'random variable' is used in probability theory. Indeed, random operators are classes of operator-valued random variables. There is in general no randomness assumed in the potential. For example, if T is a translation on the torus, the random operators are quasiperiodic. Using Pastur-Kotani theory, Conjecture **??** enlarges the class of ergodic discrete one-dimensional random Schrödinger operators L, for which there is no absolutely continuous spectrum.

Conjecture **??** implies with [111] that for a Baire generic ergodic $T \in \mathcal{X}$, the corresponding ergodic operators

have no absolutely continuous spectrum for large λ. [2] For non-ergodic T, the absolutely continuous spectrum is only absent for a large set of $x \in \mathbf{T}^d$ because the Lyapunov exponent can be zero on a set of positive measure. Indeed, for invariant measures on KAM tori, the corresponding ergodic almost periodic Schrödinger operator can have some absolutely continuous spectrum for small λ [78]. [3] The size of the spectrum, where the Lyapunov exponent vanishes can be estimated if the orbits do not stay too long at the same x value for a long time [138].

7) **Discrete Spectrum**.
The question of localization of the operators $L_{\lambda \cos}$ [134] stays open. The conjugation of the map to a Bernoulli shift on some positive Lebesgue measure suggests that eigenvalues should exist for similar reasons as in the case of independent, identically distributed potentials.



Wiener test for localisation for $\lambda \in [0, 10]$. To the left, the Mathieu case (n=100'000), to the right, the Chiricov case (n=10'000 averaged over 50x50=2500 orbits).

In the Figure to the left, numerical results are shown for the Fourier coefficients $\hat{\mu}_n = (\phi, \exp(-inL)\phi) = \int_{\mathbf{T}} e^{-in\theta}\, d\mu_\phi(\theta)$ of the spectral measure μ_ϕ where $\phi = (\dots, 0, 1, 0, \dots) \in l^2(\mathbf{Z}, \mathbf{C})$. We see the case of the Standard map operator and Mathieu operator. With a Wiener criterion for the existence of discrete spectrum, (where we compute a fast discrete quantum evolution using a related operator which has the same type of spectral measures [76]), we find numerical evidence for some point spectrum for $L(x, y)$ for a set of $(x, y) \in \mathbf{T}^2$ of positive measure, if λ is large.

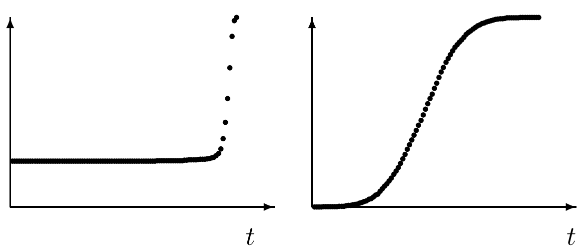
In the picture to the left, we confirm numerically the theoretically established localization transition for the Mathieu operator at $\lambda = 2$ [60, 61]. In the picture to the right we see an indication for some point spectrum in the Chirikov case for $\lambda > 0$.

Note that in the same way as for the Anderson operator, the operator $L_T(x, y)$ has for an Anosov T purely singular continuous spectrum for a Baire generic set of (x, y). This can be derived from general principles [130]. Also, for Baire generic $(T, (x, y)) \in \mathcal{X} \times \mathbf{T}^2$ and $\lambda > \lambda_0$, one knows that $L_{T,\lambda \cos}(x, y)$ has purely singular continuous spectrum. This means that $L(x, y)$ on $l^2(\mathbf{Z}, \mathbf{C})$ has Baire generically no bound states and no extended states.

8) **The distribution of the Lyapunov exponents**.

[2] The space $\mathcal{X}$ becomes a complete metric space with the metric $\rho(T, S) + \rho(T^{-1}, S^{-1})$.
[3] The results in [78] have not been written down in a final form since both of us got engaged in other projects.

$m(\{(x,y) \mid X^*_\lambda(x,y) \leq t\})$



The distribution functions of the random variable X^*_λ for $\lambda = 2, 10$. For $\lambda = 2$, $m(X_\lambda = 0) > 0$ (KAM).

For every λ, the pointwise Lyapunov exponent $(x,y) \mapsto \mu_\lambda(x,y)$ is a random variable on $\mathbf{T}^2$ which is T-invariant and has a mean $\geq \log(|\lambda|/2) - C(\lambda)$. The numerical experiments indicate that the random variables $X_\lambda(x,y) = \mu_\lambda(x,y)$ have the property that the normalizations $X^*_\lambda = (X_\lambda - \mathrm{E}[X_\lambda])/\sqrt{\mathrm{Var}[X_\lambda]}$ converge in distribution for $\lambda \to \infty$. The numerical results indicate that such a central limit theorem might indeed hold. While the tools used here for estimating $\mathrm{E}[X_\lambda]$ do not give information about the distribution, it is reasonable that the randomness in the map assured by Pesin theory makes the Lyapunov exponents in the limit $\lambda \to \infty$ behave like the Lyapunov exponents of Markovian cocycle, for which central limit theorems are known [5, 13].

9) **The size of the Pesin region**.
Because the averaged Lyapunov exponent of the Standard map can not be bigger than $\log(\lambda/2) + C(2,\lambda)$, the conjectured bound would lead to the size of the Pesin region, the set where the pointwise Lyapunov exponent of the Standard map is positive, has Lebesgue measure which is bounded below by

$$\frac{(\log(\lambda/2) - C(\lambda))}{(\log(\lambda/2) + C(2,\lambda))} = \frac{(\log(\lambda/2) - \mathrm{arcsinh}(1/\lambda) - \log(2/\sqrt{3}))}{(\log(\lambda/2) + C(2,\lambda)} . \tag{10}$$

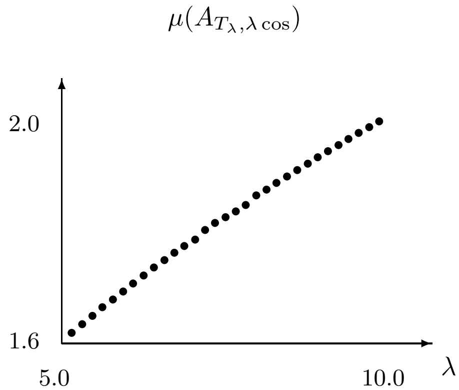
For example, for $\lambda = 5.42$, the Pesin region would already cover more than half of the phase space. (In comparison, it was measured numerically that for $\lambda = 5.0$, the size of the elliptic regions is less than 2 percent of the phase space [19]). It would follow from formula (10) that the set, where the Lyapunov exponent is zero has measure which is smaller than $O(\log(4/3)/\log(\lambda))$ for $\lambda \to \infty$. If $C(\lambda)$ could be replaced by $c(\lambda) = m(2,\lambda) = \int_{\mathbf{T}^2} \log ||dT_{\lambda \cos}(x,y)||\, dxdy = O(1/\lambda)$ the complement of the Pesin region would be the order $O(1/(\lambda \log(\lambda)))$.
Empirically, the set with stable behavior has been measured in [19]. Empirical formulas have to be taken with a grain of salt for large λ because the size of the elliptic islands is usually so small that say for $\lambda = 100$ already, a computer can hardly resolve individual islands. We know that from formula (10) that the total measure must then be less then 3/1000. The size of the elliptic islands of a periodic orbit with period p is expected to be of the order M^{-3}, where $M = \sup_{j<p} ||dT^j(x)||$ [16]. There are examples in Hamiltonian dynamics where it is possible to prove that elliptic regions cover a substantial part of the phase space [108]. For other nonergodic Hamiltonian flows, see [30]. The Carleson problem is the question whether there are λ for which $T_{\lambda \sin}$ is ergodic.

10) **Vlasov-Toda deformation of Twist maps**.
Other cocycles with the same bounds on the Lyapunov exponent can be obtained with isospectral Vlasov-Toda deformations of the random Jacobi operators $L = \Delta + V = a\tau + (a\tau)^* + b$ [71]. Assume we deform L and so the transfer cocycle with the first Toda flow $\dot{a} = a(b(T) - b), \dot{b} = 2a^2 - 2a^2(T^{-1})$, where

$$\begin{aligned} a(x,y) &= h_{12}(x,y) = \partial_x \partial_y h(x,y) \\ b(x,y) &= h_{11}(x,y) + h_{22}(T^{-1}(x,y)) = \partial_x \partial_x h(x,y) + \partial_y \partial_y h(T^{-1}(x,y)) . \end{aligned}$$

It is not clear whether the deformed operator $L_t = a_t\tau + (a_t\tau)^* + b_t$ is the Hessian of a critical point defining a twist map. The question is whether a generating function h can be derived from the deformed a, b such that h generates a twist map T satisfying discrete time Euler-Lagrange equations $h_2(T^{-1}) + h_1 = 0$. Because the deformation keeps the Lyapunov exponents of the cocycle constant, the lower bound on the entropy would hold also for the deformed twist maps. In any case, even if the 'isospectral deformation of the twist maps' should not exist in general, Toda deformation gives infinite dimensional manifolds of cocycles over a fixed $T \in \mathcal{Y}$ obtained from the cocycle $A_{T,\lambda \sin}$ of the Standard map for which one has the same Lyapunov exponent.

11) **Continuity of the entropy**.
The map $\lambda \mapsto \mu(T_{\lambda \sin})$ is upper-semicontinuous. We don't know whether it is continuous. On the other hand, the topological entropy depends continuously on λ because it is always a continuous function on C^∞ diffeomorphisms on two-dimensional manifolds [110].



The topological entropy of surface diffeomorphisms changes only through homoclinic bifurcations [151]. Motivated from results on unimodal maps [31] and consistent with experiments, it is reasonable to ask whether the topological entropy or the metric entropy with respect to the invariant Lebesgue measure depends monotonically on λ. The Figure to the left shows the numerically computed Kolmogorov-Sinai entropy for 30 values of λ in the interval $[5.0, 10.0]$ (Lyapunov exponents averaged over 40×40 orbits each of length 10^6.)

The dependence of the entropy on λ even appears to be smooth. Oscillations indeed become small for larger λ because the established lower bound is realanalytic in λ and the effective value of the entropy is in a small corridor $\log(\lambda/2) + [-c(\lambda), c(\lambda)]$ which becomes increasingly narrow for $\lambda \to \infty$.

12) **Nonuniform hyperbolicity and homoclinic tangencies**.
Assume λ is such that T_λ has positive Lyapunov exponents on a set of positive Lebesgue measure. The Pesin region of positive Lebesgue measure is contained in the closure of transverse homoclinic points or in the closure of hyperbolic periodic orbits with transverse homoclinic intersections (see [65]). We don't expect however in the case of the Standard map that for some λ, we have uniform hyperbolicity on a set Y of positive Lebesgue measure. This is supported by the fact that for large λ, there exists a dense set of parameter values λ for which homoclinic bifurcations [114] happen in the Standard map family [32]. Uniform hyperbolicity on $\mathbf{T}^2$ is excluded because the Chirikov Standard map can not be Anosov on the whole torus $\mathbf{T}^2$: it would be topologically conjugated to a hyperbolic automorphism on the torus [96] and by homotopy to $(x, y) \mapsto (2x - y, x)$ which is not hyperbolic. Equivalent to uniform hyperbolicity on a set $Y \subset \mathbf{T}^2$ is the property that there exists an interval I containing $E = 0$ such that I is disjoint from in the spectrum of $L(x, y)$ for almost all $(x, y) \in Y \subset \mathbf{T}^2$. This is equivalent to the fact that the random operator L associated to an invariant ergodic set of positive Lebesgue measure is invertible in the corresponding crossed product algebra. It is an open question whether hyperbolic sets always have measure zero or one in the Chirikov Standard map case. This property is Baire generic in the class of measure preserving C^1-diffeomorphisms [95] and probably always holds for realanalytic diffeomorphisms on the torus. Having excluded uniform hyperbolicity almost everywhere, the positive entropy result could provide a new proof that homoclinic tangencies and consequently elliptic islands occur in the Chirikov Standard map for a dense set of parameters in $[\lambda_0, \infty)$.

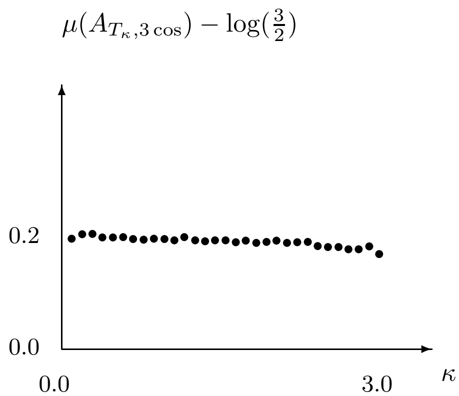
13) **Coexistence**.
Conjecture 1.2 establishes the existence of explicit real analytic maps on the torus, where coexistence of elliptic islands and positive metric entropy holds. Previously known were piecewise smooth maps [146, 147] or C^∞ maps [119]. Whether true coexistence (in the sense that the Pesin region as well as its complement are dense [137]) can hold on open sets of the phase space for some Standard maps is not known. According to [137], this coexistence problem was posed already before 1969 by Sinai. It was measured numerically in [141] that the closure of some orbits might have fractional box counting dimension. In higher dimensions, Mather conjectured [100] that generically one should have transitivity in the sense that there exists orbits which are dense in the phase space. No mathematical results about these questions are yet available. It has also been conjectured that transitive components in the complement of quasiperiodic sets are trellises [34], closed sets obtained by closing the unstable manifold of some hyperbolic periodic point. For the Standard map, where we know now the existence of ergodic sets of positive measure, one can ask about the nature of these ergodic sets (i.e. the Hausdorff dimension of the closure). Careful measurements done in [101] indicate that some ergodic components are obtained as the closure of some aperiodic orbits.

14) **The complement of the Pesin region**.
An interesting question is what is left on the complement of the union of the almost periodic KAM set and the Pesin region. Is the dynamics weakly mixing but not mixing on a set of positive measure? While Pesin theory together with a genericity result for shift invariant measures [77] provide many invariant measures which have this property, they are in general supported on sets of zero Lebesgue measure. This observation by [6] can be strengthened by estimating the Hausdorff dimension of the measure. It was first observed in [18] that there are invariant measures μ_λ whose Hausdorff dimension goes to 2 for $\lambda \to \infty$. This observation is based on Young's formula [153] for the Hausdorff dimension $H(\mu)$ of μ:

$$H(\mu) = 2h_\mu(T)/\lambda(\mu) ,$$

where $\lambda(\mu)$ is the Lyapunov exponent integrated with respect to the measure μ and where $h_\mu(T)$ is the metric entropy of T with respect to the measure μ.
While periodic orbits are dense on the Pesin region [64] and on the closure of KAM orbits (in the sense that for x in the KAM set, there exists periodic $y_n \to x$), the question whether periodic orbits are dense in $\mathbf{T}^2$ is open and related to the problem to analyze the dynamics on the complement of the union of the Pesin and KAM regions.

15) **A variational problem**.



For fixed λ, is there (a realanalytic) $T \in \mathcal{X}$ (rsp. $\mathcal{Y}$) for which the Lyapunov exponent of the fixed matrix valued map $A_{\lambda\cos}$ is maximal? If T is the identity, we have $\mu(A_{\lambda\cos,T}) \geq \log|\lambda/2|$. There are many maps $R_\alpha T R_{-\alpha}$ with $R_\alpha(x,y) = (x+\alpha,y)$ for which the Lyapunov exponent is bigger than $\log(\lambda/2)$. If we fix the cocycle A_λ, how does the Lyapunov exponent depend on T_κ, when we deform T_κ from the Mathieu case T_0 to the Standard map case T_λ? The graph to the left shows a numerical computation in the case $\lambda = 3.0$. The averaged Lyapunov exponent seems to decrease slightly when κ is increasing.

The fact that the Lyapunov exponent is close to the one in the Mathieu case $\kappa = 0$ is compatible with the empirical fact of being close to Aubry duality (see 3) 25)).
It would be good to know the Taylor expansion of

$$\kappa \mapsto \int_{\mathbf{T}^2} \log([A^n_{E,\lambda,T_\kappa}(w)(x,y)]_{11}) \; dxdy$$

at $\kappa = 0$, for $|w| < 1$ which exists and converges for large λ. This requires to understand the dynamics for complex κ, where (x_n, y_n) become complex too.

16) **Thermodynamic limit**.
For classes of symplectic coupled map lattices like finite dimensional versions of [63], we could obtain lower bounds for the entropy which are independent of the dimension of the map. In the thermodynamic limit, these maps define homeomorphisms T on the compact metric space $(\mathbf{T}^2)^{\mathbf{Z}}$ which preserve the product measure. While one has no more an Oseledec theorem in this infinite dimensional situation (the results in [126, 93] do not apply), the averaged maximal Lyapunov exponent $\lim_{n\to\infty} n^{-1} \int_{(\mathbf{T}^2)^{\mathbf{Z}}} \log ||dT^n(\mathbf{x}|| \; d\mathbf{x}$ is defined and positive for large λ and small ϵ. The cocycle $A = dT$ is now a bounded linear operator on $l^2(\mathbf{Z}, \mathbf{C}^2)$. By Vesantini's theorem [121], the map $w \mapsto n^{-1}\log(\rho(A^n(w))$ is subharmonic if $\rho(A)$ denotes the spectral radius of A. The estimates can be done directly for the operator-valued cocycle. It is not clear however whether the metric entropy of the homeomorphism T with invariant product measure is the limit of the metric entropies in the finite-dimensional

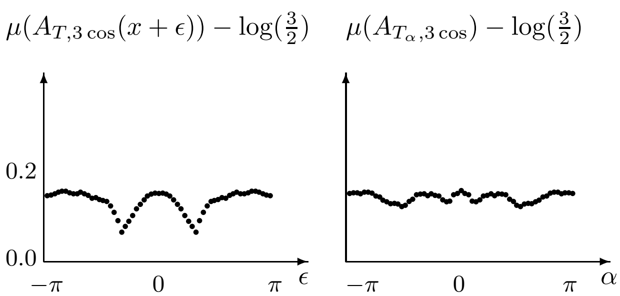
situations.

17) **Response formulas**.
The subharmonic estimates could apply for classes of dissipative Standard maps on the cylinder. We expect the averaged Lyapunov exponent to depend analytically on the map in open sets of realanalytic maps because our results imply that smooth observables see a pretty uniform hyperbolic behavior of the dynamics. Indeed, by taking periodic Rohlin approximations to the map T, we get

$$n^{-1} \int_{\mathbf{T}^2} \log ||dT^n_{\lambda \sin}|| \; dxdy \geq \log(\lambda/2) - C_n(\lambda) \; ,$$

where $C_n(\lambda) = C(\lambda) + (\log(\lambda/2) + C(\lambda))/n$. The hyperbolicity appears to be uniform when observing the dynamics with smooth observables.
Could it be possible even that Ruelle's response formula [128] holds for some Standard maps with positive Lyapunov exponents? This formula says that for an infinitesimal perturbation $T + \delta T$ of an Anosov map T, the natural T-invariant measure ρ_T will change to $\rho_{T+\delta T} = \rho_T + \delta\rho_T$ with $\delta\rho_T(\Phi) = -\sum_{n=0}^{\infty} \rho_T((\Phi \circ T^n) \cdot \mathrm{div}(X))$, where $X = \delta T \circ T^{-1}$ is the vector field associated to the change of the map and where $\Phi \in C(\mathbf{T}^d)$ is an observable. The response formula would be expected to hold only for real analytic perturbations of T in $\mathcal{X}$ and for a subset of smooth observables Φ which do nowhere vanish on the torus. One has to see this question in the more general context of the 'chaotic hypothesis' [42].

18) **Symmetry**.
The symmetry $T(-x,-y) = -T(x,y), DT(x,y) = DT(-x,-y)$ which holds in the case of the Chirikov Standard map seems to make the Lyapunov exponent extremal with respect to some parameter perturbations. This is supported by some numerical experiments. It is not known whether the Lyapunov exponent is continuous as a function of w for example when $w \mapsto L_w$ is the random operator (1), so that symmetry does not imply a local extremum.



Computing the Lyapunov exponent in two cases with lack of symmetry. To the right, we use $T_\alpha(x,y) = (2x + 3\sin(x) - y + \alpha, x) \in \mathcal{X}$. The estimate known for $\epsilon = \alpha = 0$ extend.

In the Figure to the left, we see two experiments. The left graph shows the Lyapunov exponent of the deformed cocycle $\phi \mapsto A(x + \phi)$, where $A(x)$ is the Jacobean cocycle of the Standard map. This is equivalent to a deformation of the map T by conjugation $T_\phi = R_\phi^{-1} \circ TR_\phi$. In the right graph, the map T is moved along a path $\alpha \mapsto T_\alpha = R_\alpha \circ T$ in $\mathcal{X}$, where $R_\alpha(x,y) = (x + \alpha, y)$. For $\phi = \alpha = 0$, we have the Chirikov Standard map case with $\lambda = 3$. In the graph to the left, Herman subharmonicity argument shows that $\int_{\mathbf{T}} \mu(A_{T,3\cos}(\cdot + \alpha)) \; d\alpha - \log(3/2) > 0$. Nevertheless, in both experiments, we never see a Lyapunov exponent below $\log(\lambda/2)$.

19) **Diffusion and Sinai's (H1) conjecture**.
A discrete Legendre transform brings the Standard map into the Hamiltonian form $T : (x,y) \mapsto (x + y + f(x), y + f(x))$ which is a map on the cylinder, the cotangent bundle $T^*\mathbf{T} = \mathbf{T} \times \mathbf{R}$ of the circle. Let $A \subset \mathbf{T}^2$ be a measurable T-invariant set of positive Lebesgue measure. If $(x_j, y_j) = T^j(x,y)$ is an orbit in the universal cover $\mathbf{R}^{2d}$, then $X_j = y_j - y_{j-1} = f(x_j) = \psi(x_j, y_j)$ are random variables on the probability space Ω equipped with the normalized Lebesgue measure. They have the mean $\mathrm{E}[X_j] = \int_{\mathbf{T}^2} f(x_j) \; dxdy = 0$ and the variance $\mathrm{Var}[X_j] = \int_{\mathbf{T}^2} \psi(x,y)^2 \; dxdy$. Interesting is the growth rate of $S_n^2 = (\sum_{j=0}^{n-1} X_j)^2 = (y_n - y_0)^2$. Using

translational invariance $\mathrm{E}[X_j X_l] = \mathrm{E}[X_{j+1}, X_{l+1}]$, the variance of S_n is

$$\mathrm{Var}[S_n] = \mathrm{E}[(\sum_{j=0}^{n-1} X_j)^2] = n\mathrm{Var}[X] + \sum_{j=1}^{n-1}(n-j)\mathrm{E}[X_0 X_j] = n\mathrm{Var}[X] + \sum_{j=1}^{n-1}(n-j)\hat{\mu}_\psi(j) ,$$

where μ_ψ is the spectral measure of $\psi \in L^2(\Omega), \psi(x,y) = f(x)$ with respect to the unitary Koopman operator $g \mapsto g(\tilde{T})$, where $\tilde{T}$ is the map T induced on Ω. The Fourier transform of μ_ψ is $\hat{\mu}_\psi(j) = \mathrm{E}[X_0 X_j] = \mathrm{Cov}[X_0, X_j]$, a correlation function. Let β be the infimum over all real numbers for which $\limsup_{n\to\infty} n^{-\beta} \sum_{j=1}^{n-1}(n-j)\hat{\mu}_\psi(j)$ is finite. If $\beta = 1$, then S_n behaves like a random walk (a case, where the X_j are independent) and $D = \int_{\mathbf{T}^2} V'(x)^2 \, dx + \limsup_{n\to\infty} n^{-1} \sum_{j=1}^{n}(n-j)\hat{\mu}_\psi(j)$ is the diffusion constant.
It is a conjecture of Sinai (H1) on page 144 in [131] that there exists a set Ω of positive Lebesgue measure for which $\beta = 1$ if λ is larger then λ_{crit}, where the last homotopically nontrivial KAM torus disappears. If $\hat{\mu}_\psi(n)$ would decay fast enough, then $D = \int_{\mathbf{T}^2} V'(x)^2 \, dx + \sum_{j=0}^{\infty} \hat{\mu}_\psi(j)$. First numerical experiments were done by Chirikov and Hizanidis [122]. Numerically, the Standard is reported to show such Brownian diffusive behavior for large λ [86, 83]. The fact of having positive Lyapunov exponents on a set of positive measure makes it plausible that the random variables X_0, X_n get decorrelated for $n \to \infty$. When computing numerically the first few hundred Fourier coefficients for smaller g (like $g = 5$) and checking with the Wiener theorem, we got the impression that μ_f still has some atoms (which prevents decorrelation) if $\Omega = \mathbf{T}^2$. Indeed, the presence of elliptic islands could be responsible for an almost periodic component in the Fourier transform of μ_f. In numerical experiments, one can get rid of this discrete part of the spectrum by adding stochastic noise [86].
In any case, the (H1) conjecture of Sinai would be settled for $\lambda > \lambda_0$ if one could show a fast enough decay of correlation of the spectral measure of $\psi(x,y) = f(x)$ on a mixing component of the Pesin set Ω. No exponential decay is necessary. It is enough to establish a power law decay of correlation $\hat{\mu}_\psi(j) = O(j^{-2})$. This finite differentiability condition for the spectral measure μ_ψ is reasonable since the dynamics on Ω is conjugated to a Markov chain. We even expect many spectral measures to be realanalytic leading to exponential decay of correlations.

20) **Dissipative Standard maps**.
The Lyapunov exponent of dissipative Standard maps numerically often satisfy the lower bound of the conservative case. The Lyapunov exponent can drop however to zero. In order that the proof carries over to the dissipative case, it appears however, that the random variables x_j need a smooth distribution μ sufficiently close to the uniform distribution. Results for Henon maps lead to the expectation that for many parameters, there exist invariant SRB measures for dissipative Standard maps like

$$T_b : (x,y) \mapsto (x + y + \lambda\sin(x), b(y + \lambda\sin(x))) ,$$

with $b < 1$. Results like in [15, 103] support this. Viana conjectured that in general a map with nonzero Lyapunov exponents almost everywhere in the phase space has an SRB measure [142]. If this conjecture is true, it indicates that for most values of b, an SRB measure should exist.
The estimates of Lyapunov exponents using spectral methods could even carry over to the case $b = 0$, where on each invariant set $y = \alpha$ we get the one-dimensional Arnold family $x \mapsto x + \alpha + \lambda\sin(x)$. If such a circle map has a smooth invariant measure sufficiently close to the Lebesgue measure, the Lyapunov exponent with respect to this measure should be $\geq \log(\lambda/2) - C(\lambda)$ for large λ.

21) **More general stability of positive metric entropy?**
The stability, we have established holds for estimates of the entropy which are done using subharmonicity rsp. the Jensen formula. This result provokes the question, whether, in general, positive metric entropy is an open property in the realanalytic category: is it true that for any realanalytic, measure preserving diffeomorphism T on the torus with positive metric entropy, there exists a Banach space of realanalytic measure-preserving maps such that an open neighborhood of T has positive metric entropy?
A question related to this stability problem is whether the Riesz measure dk of the w-parameterization of the operator L for a general twist map has the property that the potential $\int \log|w - w'| \, dk(w')$ does not fluctuate too much around its mean on the unit circle $\{|w| = 1\}$. This could be used to establish the stability if the entropy is comparable to upper bounds of the entropy like in the Standard map case.

22) **The Herman spectrum**.
One can also look at different analytic parameterizations of the cocycle. The Herman spectrum of a cocycle A is the set of complex numbers z, such that $R(z)A$ is not uniformly hyperbolic, where $R(z) = \begin{pmatrix} \frac{z+z^{-1}}{2} & \frac{z-z^{-1}}{2i} \\ -\frac{z-z^{-1}}{2i} & \frac{z+z^{-1}}{2} \end{pmatrix} \in SL(2,\mathbf{C})$ which has the property that $R(e^{i\alpha}) = R(\alpha) \in SO(2,\mathbf{R})$. It is a subset of $\{|z| = 1\}$ because for $|z| \neq 1$, one can find a strict coinvariant cone field in $\mathbf{C}P^1$. There is a measure μ supported by the Herman spectrum and a harmonic function g such that the Lyapunov exponent $\mu(z) = \mu(zR(z)A)$ satisfies

$$\mu(z) = \int \log|z - z'| \, d\mu(z) + g(z) \; .$$

This abstract Thouless formula [56] follows from Riesz theorem and μ is the Riesz measure of the subharmonic function $\mu(z)$. The proof of Conjecture **??** shows $\log(\mu(\beta)) > \log(\cos(\beta)\lambda/2) - O(1/\lambda)$ for all β. The Lyapunov exponent is realanalytic and positive outside the Herman spectrum. At such points, the directional derivative with respect to variations of the angle β in $z = re^{i\beta}$ can be computed using a formula of Ruelle [124] for the Fréchet derivative of Lyapunov exponents on the open set of uniformly hyperbolic cocycles. One gets $\frac{d}{d\beta}\mu(\beta) = \int_{\mathbf{T}^2} \cot(\omega(x,y)) \, dm(x)$, where $\omega(x)$ is the angle between the stable and unstable directions m^+, m^- at the point (x,y) (see [72]). This formula shows that the Lyapunov exponent can change a lot if the stable and unstable manifolds are close. In the case $A(x) = \begin{pmatrix} c & b(x) \\ 0 & c^{-1} \end{pmatrix}$ with constant $|c| \neq 1$, one obtains

$$\frac{d}{d\beta}\mu(A(\beta)) = \int_{\mathbf{T}^2} \cot(\omega(x,y)) \, dxdy = \frac{c}{1-c^2} \int b \, dxdy \; .$$

If A is the cocycle of a Standard map in Hamiltonian form like in 19), there is a spectral gap containing $\beta = -\pi/4$ in the Herman spectrum, because $R(-\pi/4)dT_\lambda = \begin{pmatrix} 2^{-1/2} & 0 \\ 2^{-1/2} + 2^{1/2}\lambda \cdot \cos(x) & 2^{1/2} \end{pmatrix}$. We compute $\frac{d}{d\beta}A(-\pi/4) = 1$. The point $z = e^{i0}$ is in the Herman spectrum because it follows from [97, 127] (see also [72]) that $z = 1$ is outside the Herman spectrum if and only if the map $T_{\lambda f}$ is Anosov. The rotation number of Ruelle [127] defined for lifts of A into the universal cover of $\mathrm{SL}(2,\mathbf{R})$ plays the role of the integrated density of states in the case of the Schroedinger spectrum. It is uniquely defined up to a multiple of 2π if one fixes $\rho(A(0)) = 0$. The rotation number is constant on an interval I if and only if I is not in the Herman spectrum [72]. Furthermore, $\rho(\beta) = \int_0^\beta \, d\mu(\alpha)$, showing that μ is an 'integrated density of states'. From the stable and unstable direction fields, one can construct (nonselfadjoint) random Jacoby operators $L(z)$ having those direction fields $m^\pm(z)$ as Titchmarsh-Weyl functions (with energy $E = 0$). They are analytic in z outside the Herman spectrum. Call a subset of the unit circle a part of the 'absolutely continuous spectrum' of $L(z)$ if there, $m^+(z) = \overline{m^-}(z)$. This absolutely continuous spectrum is a subset of the Herman spectrum. An adaption of Kotani theory [23] shows that the 'absolutely continuous spectrum' is the essential closure of the set, where the Lyapunov exponent is zero and that in the ergodic case, the existence of some absolutely continuous spectrum implies that T is 'deterministic' (see [40, 85, 80]). We know therefore for the Standard map that there is no absolutely continuous Herman spectrum for almost all (x,y) in the Pesin region if λ is large. We call z an 'eigenvalue' of $L(z)(x,y)$ if $L(z)(x,y)$ has an eigenvalue $E = 0$. The set of these 'eigenvalues' forms a 'discrete spectrum' of $L(z)(x,y)$ which is a subset of the Herman spectrum and which we expect to be nonempty for Lebesgue almost all (x,y) in the Pesin region.

24) **Quasiconformality**.
The map

$$z \mapsto G_n(z) = \int_{\mathbf{T}^1} \log|z^n A_E^n(z,y)| \, dy$$

is not conformal for small $|z|$ in general. For realanalytic T near a linear automorphism $(x,y) \mapsto (x+y,y)$ it is quasiconformal for all n. Question: is there an open set of realanalytic, measure-preserving maps T for which G_n are quasiconformal? If yes, it would be important to estimate the complex dilation $\kappa(z) = \overline{\partial}G_n(z)/\partial G_n(z)$ and to estimate the Lyapunov exponent in terms of κ. For $\kappa = 0$, one has conformality and the subharmonic estimates of Herman apply.

25) **The Aubry duality transform**.
The Aubry duality transform is an involutive map on a class of operators. It provides in the Mathieu case an elegant way to estimate the Lyapunov exponent. The transform can be defined in more general situations: Let T_k be a cyclic interval exchange transformation on $\mathbf{T}$ and let $L_{\lambda \cos, T_k}$ be the corresponding Schrödinger operator. The density of states of the random operator L is the same as the density of states of the operator

$$(Lu)_n(\theta) = u_{n+1}(\theta) + u_{n-1}(\theta) + \lambda \cos(T^n \theta) u_n(\theta)$$

on the Hilbert space $H = L^2(\mathbf{T} \times \mathbf{Z})$. There is a piecewise smooth potential V such that $\cos(T^n(\theta)) = V(\theta + n\alpha)$, where $\alpha = 1/k$. Let $V(\theta) = \sum_n V_n \exp(in\theta)$ be its Fourier series. The duality transform (see [50])

$$(Uu)_m(\eta) = \sum_{n \in \mathbf{Z}} \int_{\mathbf{T}} e^{-(\eta + 2\pi m\alpha)n} e^{-im\theta} u_n(\theta) \, d\theta$$

satisfies $\tau U = (\lambda/2) U \sigma$, where $\tau u_n(\theta) = u_{n+1}(\theta)$ and $\sigma u_n(\theta) = \exp(i(\theta + n\alpha)) u_n(\theta)$. The operator L can be written as $L = \tau + \tau^* + \lambda \sum_n V_n \sigma^n$ so that $U^* L U = \sigma + \sigma^* + \lambda/2 \sum_n V_n (\tau^n + (\tau^n)^*)$. We are interested in the density of states dk_ϵ of $L_\epsilon = (2/\lambda) U^* L U = \epsilon(\sigma + \sigma^*) + \sum_n v_n (\tau^n + \tau_n^*)$ and $f(\epsilon) = \log(\det(L_\epsilon)) = \int \log|E| \; dk_\epsilon(E)$ which satisfies $f(0) = 0$. Because $\log(\det(L_{\lambda \cos, T_k})) \geq \log(\lambda/2) + f(2/\lambda)$, we want to estimate $f(\epsilon)$ from below for small ϵ. If $T_k(x) = x + 1/k$, where $f(\epsilon)$ is the Lyapunov exponent of a symplectic transfer cocycle, we have $\int \log|E| \; dk_\epsilon(E) \geq 0$. In general, the Thouless formula just becomes $\log(\det(L_\epsilon)) = \lim_{N \to \infty} \sum_{k=1}^{2N} \int_{\mathbf{T}} \log(\lambda_j(x, \epsilon)) \; dx$, where $\lambda_j(\epsilon, x)$ are the eigenvalues of the truncated $N \times N$ matrix $L_{\epsilon,N}(x)$. While $f(0) = 0$ and a perturbation lemma of Lidskii ([129]) assures that $|\lambda_j(\epsilon, x) - \lambda_j(\epsilon, 0)| \leq \epsilon$, this does not allow us to estimate $f(\epsilon)$ from below. The perturbation problem to estimate $f(\epsilon)$ seems still difficult and it is not clear, whether the Aubry transform, which transformed the perturbation problem $V \mapsto V + \epsilon\Delta$ into a perturbation $\Delta \mapsto \Delta + \epsilon V$ has made things simpler.

26) **Lower bound for topological entropy?**
The topological entropy of $T \in \mathcal{X}$ on $\mathbf{T}^2$ is bounded below by $\log(\mathrm{sp}(T_*))$, where $\mathrm{sp}(T_*)$ is the spectral radius of $T_* : H_*(\mathbf{T}^2) \mapsto H_*(\mathbf{T}^2)$ [102]. Can the metric entropy of a (smooth) $T \in \mathcal{X}$ with respect to the invariant Lebesgue measure become smaller than $\mathrm{sp}(T_*)$?

27) **Conjectures on the Standard map**.
The following of conjectures about Standard maps have still to be settled.

I) In [19], the entropy of the Standard map $T_{\lambda \sin}$ was measured $\geq \log(\lambda/2)$. Chirikov formulated the possibility that the elliptic islands might cover arbitrary large regions of the phase space.

I [72]: The Kolmogorov-Sinai entropy of the Chirikov Standard map is $\geq \log(\lambda/2)$.

II) [134] introduced the problem of determining the spectrum of random operators $(L(x,y)u)_n = u_{n+1} - 2u_n + u_{n-1} + \lambda \cos(x_n) u_n$, where $T^n(x,y) = (x_n(x,y), y_n(x,y))$ is the orbit starting at (x,y).

II [134]: The operator $L(x,y)$ has some point spectrum for a set (x,y) of positive Lebesgue measure if λ is large enough.

III) [16] asked whether there exists a set of parameters λ with full density at ∞ for which the Chirikov Standard map has no elliptic islands. We know that we have for large λ an open dense set of parameters with no ergodicity and positive entropy.

III [16]: There are parameters λ with full density at ∞ for which the Chirikov Standard map $T_{\lambda \sin}$ is ergodic.

Heuristic arguments in [47] predict that the Lebesgue measure of the set of parameters $\lambda \in [r, r+1]$ which lead to nonergodic Standard maps $T_{\lambda \sin}$ is of the order $O(1/r)$.

IV) The following problem is related to III):

IV [132]: For large λ, the Standard map has (in some Baire topology of maps) a neighborhood in which a residual set of f's gives ergodic maps T_f.

V) In the textbook [131] p. 144, conjecture (H2) contained as a second part the statement that the entropy grows to infinity for $\lambda \to \infty$. While we solved this part of the problem here, the first part of the conjecture is still open:

V [131]: the entropy of the Chirikov Standard map is positive for all $\lambda > 0$.

One should be able to prove that for any real-analytic non-constant periodic f, there exists $\lambda_0 > 0$ such that the Standard map $T_{\lambda f}$ has positive Kolmogorov-Sinai entropy for all $\lambda > \lambda_0$ and that for any real-analytic, non-constant, periodic map $f : \mathbf{T}^d \to \mathbf{R}^N$ and every symmetric, constant matrix $E \in GL(N, \mathbf{Z})$, all the Lyapunov exponents of the symplectic map $T_{Ex+\lambda f}$ are nonzero on a set of positive Lebesgue measure for large enough $|\lambda|$. All these properties are expected to be stable with respect to realanalytic perturbations of the map $T_{Ex+\lambda f}$, and hold for a fixed cocycle for all $T \in \mathcal{Y}$.

References

[1] M. Aigner and G.M. Ziegler. *Proofs from the book.* Springer Verlag, Berlin, 1998.

[2] S. Alpern and V.S. Prasad. Combinatorial proofs of the Conley-Zehnder-Franks theorem on a fixed point for torus homeomorphisms. *Adv. in Math.*, 99:238–247, 1993.

[3] M. Andersson. *Topics in complex analysis.* Universitext. Springer-Verlag, New York, 1997.

[4] S. Angenent. Monotone recurrence relations their Birkhoff orbits and topological entropy. *Ergod. Th. Dyn. Sys.*, 10:15–41, 1990.

[5] L. Arnold, W. Kliemann, and E. Oeljeklaus. Lyapunov exponents of linear stochastic systems. In *Lyapunov exponents (Bremen, 1984)*, pages 85–125. Springer, Berlin-New York, 1986.

[6] S. Aubry. The concept of anti-integrability: definition, theorems and applications to the Standard map. In K.Meyer R.Mc Gehee, editor, *Twist mappings and their Applications*, IMA Volumes in Mathematics, Vol. 44. Springer Verlag, 1992.

[7] S. Aubry and P.Y. Le Daeron. The discrete Frenkel Kontorova model and its extensions. I. Exact results for the ground-states. *Physica D*, 8:381–422, 1983.

[8] Y. Avron and B. Simon. Singular continuous spectrum for a class of almost periodic Jacobi matrices. *Bull. Amer. Math. Soc.*, 6:81–85, 1982.

[9] R. Bellman. A note on determinants and Hadamard's inequality. *Amer. Math. Monthly*, 50:550–551, 1943.

[10] C.A. Berenstein and R. Gay. *Complex variables, an introduction*, volume 125 of *Graduate texts in Mathematics*. Springer Verlag, Berlin, 1991.

[11] M. Bialy and L. Polterovich. Hamiltonian systems, Lagrangian tori and Birkhoff's theorem. *Math. Ann.*, 292:619–627, 1992.

[12] A. Bobenko, N. Kutz, and U. Pinkall. The discrete quantum pendulum. *Phys. Lett. A*, 177:399–404, 1993.

[13] P. Bougerol. Théorèmes limite pour les systèmes linéaires à coefficients Markoviens. *Probab. Th. Rel. Fields*, 78:193–221, 1988.

[14] M. Braun. Mathematical remarks on the Van Allen radiation belt: a survey of old and new results. *SIAM Rev.*, 23:61–93, 1981.

[15] P. Le Calvez. Propriétés dynamiques des difféomorphismes de l'anneau et du tore. *Astérisque*, 204:1–131, 1991.

[16] L. Carleson. Stochastic models of some dynamical systems. In *Geometric aspects of functional analysis (1989-90)*, volume 1469 of *Lecture Notes in Math.*, pages 1–12. Springer, Berlin, 1991.

[17] R. Carmona and J.Lacroix. *Spectral Theory of Random Schrödinger Operators*. Birkhäuser, 1990.

[18] J. Cheng. Horseshoes in the Standard map. Unpublished manuscript, University of Arizona, 1990.

[19] B.V. Chirikov. A universal instability of many-dimensional oscillator systems. *Phys. Rep.*, 52:263–379, 1979.

[20] B.V. Chirikov. Particle confinement and adiabatic invariance. *Proc. R. Soc. Lond. A*, 413:145–156, 1987.

[21] A. Connes. *Noncommutative geometry*. Academic Press, Harcourt Brace and company, SanDiego, 1994.

[22] W. Craig and B. Simon. Log Hölder continuity of the integrated density of states for stochastic Jacobi matrices. *Commun. Math. Phys.*, 90:207–218, 1983.

[23] H.L. Cycon, R.G.Froese, W.Kirsch, and B.Simon. *Schrödinger Operators-with Application to Quantum Mechanics and Global Geometry*. Springer-Verlag, 1987.

[24] R. de la Llave. Recent progress in classical mechanics. In *Mathematical physics, X (Leipzig 1991)*, pages 3–19. Springer, Berlin, 1992. Also document Nr. 91-73 in *mp_arc@math.utexas.edu*.

[25] F. Delyon and B. Souillard. The rotation number for finite difference operators and its properties. *Commun. Math. Phys.*, 89:415–426, 1983.

[26] M. Denker, C. Grillenberger, and K. Sigmund. *Ergodic Theory on Compact Spaces*. Lecture Notes in Mathematics 527. Springer, 1976.

[27] J. Dixmier. *Les algèbres d'opérateurs dans l'espace Hilbertien*. Cahiers Scientifiques, Fascicule XXV. Gauthier-Villars, Paris, deuxième edition edition, 1969.

[28] J. Dixmier. *Von Neumann algebras*, volume 27 of *North-Holland mathematical library*. North-Holland, Amsterdam, 1981.

[29] J. Dixon. How good is Hadamard's inequality for determinants? *Canad. Math. Bull.*, 27:260–264, 1984.

[30] V. Donnay. Elliptic ilands in generalized Sinai billiards. *Ergod. Th. Dyn. Sys.*, 16:975–1010, 1996.

[31] A. Douady. Topological entropy of unimodal maps: monotonicity for quadratic polynomials. In *Real and complex dynamical systems (Hillerod, 1993)*, volume 464 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 65–87. Kluwer Acad. Publ., Dordrecht, 1995.

[32] R. Duarte. Plenty of elliptic islands for the Standard family. *Ann. Inst. Henri Poincaré Phys. Théor.*, 11:359–409, 1994.

[33] P.L. Duren. *Univalent functions*. Springer-Verlag, New York-Berlin, 1983.

[34] R.W. Easton. Trellises formed by stable and unstable manifolds in the plane. *Trans. Am. Math. Soc.*, 294:719–732, 1986.

[35] K.J. Falconer. *The geometry of fractal sets.* Cambridge University Press, Cambridge, 1985.

[36] E. Fontich. Transversal homoclinic points of a class of conservative diffeomorphisms. *J. Diff. Equ.*, 87:1–27, 1990.

[37] C. Froeschlé. A numerical study of the stochasticity of dynamical systems with two degree of freedom. *Astron. Astroph.*, 9:15–23, 1970.

[38] B. Fuglede and R.V. Kadison. Determinant theory of finite factors. *Annals of Mathematics*, 55, 1952.

[39] A. Furman. On the multiplicative ergodic theorem for uniquely ergodic systems. *Ann. Inst. Henri Poincaré Prob. et Statist.*, 33:797–815, 1997.

[40] H. Furstenberg. Noncommuting random products. *Trans. Am. Math. Soc.*, 108:377–428, 1963.

[41] H. Furstenberg. *Recurrence in ergodic theory and combinatorial number theory.* Princeton University Press, Princeton, N.J., 1981.

[42] G. Gallavotti. Chaotic hypothesis and universal large deviations properties. pages 205–233. Extra Volume ICM 1998, I.

[43] L. Gårding. *Some points of analysis and their history.* American Mathematical Society, Providence, RI, 1997.

[44] V.G. Gelfreich. A proof of the exponential small transversality of the separatrices for the Standard map. *Commun. Math. Phys.*, 201:155–216, 1999.

[45] F. Gesztesy and R. Weikard. Floquet theory revisited. In *Differential Equations and Mathematical Physics*, pages 67–84. International Press, 1995. Proceedings of the International Conference Univ. of Alabama at Birmingham, March 13-17, 1994.

[46] D. Gieseker, H.Knörrer, and E.Trubowitz. *The Geometry of Algebraic Fermi Curves*, volume 14 of *Perspectives in Mathematics.* Academic Press, Inc, Boston, 1993.

[47] A. Giorgilli and V.F. Lazutkin. *Some remarks on the problem of ergodicity of the Standard map. mp_arc* document $99-287$, 1999.

[48] I.Ya. Goldsheid and E. Sorets. Lyapunov exponents of the Schrödinger equation with quasi-periodic potential on a strip. *Commun. Math. Phys.*, 145:507–513, 1992.

[49] C. Golé. Suspension of symplectic twist maps by Hamiltonians. In *Hamiltonian dynamical systems (Cincinnati, OH, 1992)*, volume 63 of *IMA Vol. Math. Appl.*, pages 163–169. Springer, New York, 1995.

[50] A.Y. Gordon, S. Jitomirskaya, Y. Last, and B. Simon. Duality and singular continuous spectrum in the almost Mathieu equation. *Acta Math.*, 178:169–183, 1997.

[51] G.R. Hall. Some examples of permutations modelling area preserving monotone twist maps. *Physica D*, 28:393–400, 1987.

[52] G.R. Hall. Some problems on dynamics of annulus maps. In *Hamiltonian dynamical systems (Boulder, CO, 1987)*, pages 135–152. Amer. Math. Soc., Providence, RI, 1988.

[53] P.R. Halmos. *Lectures on Ergodic Theory.* The Mathematical Society of Japan, 1956.

[54] W.K. Hayman. *Subharmonic functions*, volume 20 of *London Mathematical Society Monographs.* Academic Press, Inc. Harcourt Brace Jovanovich, Publishers, London, 1989.

[55] M.R. Herman. Some open problems in dynamical systems. *Doc. Math. J. DMV*, II:797–808. Extra Volume ICM, 1998.

[56] M.R. Herman. Une méthode pour minorer les exposants de Lyapounov et quelques exemples montrant le caractère local d'un théorème d'Arnold et de Moser sur le tore de dimension 2. *Comment. Math. Helv.*, 58:453–502, 1983.

[57] M.R. Herman. Sur les courbes invariantes par les difféomorphismes de l'anneau. Vol. 2. *Astérisque*, 144:1–248, 1986.

[58] M.R. Herman. Seminar talk given at Caltech. 1994.

[59] M.R. Herman. Stabilité topologique des systèmes dynamiques conservatifs. Undated notes, distributed at a minicourse of M. Herman given at Caltech, 1994.

[60] S.Ya. Jitomirskaya. Anderson localization for the almost Mathieu equation: A nonperturbative proof. *Commun. Math. Phys.*, 165:49–57, 1994.

[61] S.Ya. Jitomirskaya. Anderson localization for the almost Mathieu equation II, point spectrum for $\lambda \geq 2$. *Commun. Math. Phys.*, 168:201–205, 1995.

[62] K. Josic and E. Wayne. Dynamics of a ring of diffusively coupled Lorenz oscillators. *mp_arc* document $99-164$, 1999.

[63] H. Kantz and P. Grassberger. Chaos in low-dimensional Hamiltonian maps. *Phys. Lett. A*, 123:437–443, 1987.

[64] A. Katok. Lyapunov exponents, entropy, and periodic orbits for diffeomorphisms. *Pub. I.H.E.S.*, 51:137–173, 1980.

[65] A. Katok and B. Hasselblatt. *Introduction to the modern theory of dynamical systems*, volume 54 of *Encyclopedia of Mathematics and its applications*. Cambridge University Press, 1995.

[66] A. Katok and J.-M. Strelcyn. *Invariant manifolds, entropy and billiards, smooth maps with singularities*, volume 1222 of *Lecture Notes in Mathematics*. Springer-Verlag, 1986.

[67] Y. Katznelson and D. S. Ornstein. A new method for twist theorems. *J. Analyse Math.*, 60:157–208, 1993.

[68] O. Knill. The upper Lyapunov exponent of $\mathrm{SL}(2, \mathbf{R})$ cocycles: discontinuity and the problem of positivity. In *Lyapunov exponents (Oberwolfach, 1990)*, volume 1486 of *Lecture Notes in Math.*, pages 86–97. Springer, Berlin, 1991.

[69] O. Knill. Positive Lyapunov exponents for a dense set of bounded measurable $SL(2, R)-$ cocycles. *Ergod. Th. Dyn. Sys.*, 12:319–331, 1992.

[70] O. Knill. Factorisation of random Jacobi operators and Bäcklund transformations. *Commun. Math. Phys.*, 151:589–605, 1993.

[71] O. Knill. Isospectral deformations of random Jacobi operators. *Commun. Math. Phys.*, 151:403–426, 1993.

[72] O. Knill. Spectral, ergodic and cohomological problems in dynamical systems. PhD Theses, ETH Zürich, 1993.

[73] O. Knill. Random Schrödinger operators arising from lattice gauge fields I: existence and density of states. J. Math. Physics, in Press, 1999.

[74] O. Knill. Random Schrödinger operators arising from lattice gauge fields II: determinants. Old version in Mathematical Physics Preprint Archive *mp_arc* document $95-428$, 1995.

[75] O. Knill. Topological entropy of some Standard type monotone twist maps. *Trans. Am. Math. Soc.*, 348:2999–3013, 1996.

[76] O. Knill. A remark on quantum dynamics. *Helvetica Physica Acta*, 71:233–241, 1998.

[77] O. Knill. Singular continuous spectrum and quantitative rates of weakly mixing. *Discrete and continuous dynamical systems*, 4:33–42, 1998.

[78] O. Knill and Y. Last. Spectral properties of Jacobi matrices arrising from twist maps. Unpublished Manuscript, Caltech, 1994.

[79] T. Kontorova and Y.I. Frenkel. *Zhurnal Eksper. i Teoret. Fiziki*, 8:1340–1349, 1938. reference given in [7].

[80] S. Kotani and B.Simon. Stochastic Schrödinger operators and Jacobi matrices on the strip. *Commun. Math. Phys.*, 119:403–429, 1988.

[81] O. Lanford III. Introduction to hyperbolic sets. In *Regular and Chaotic Motion in Dynamical systems*, volume 118 of *Nato ASI Ser. B*, pages 73–102. 1985.

[82] P.D. Lax. Approximation of measure preserving transformations. *Commun. Pure Appl. Math.*, 24:133–135, 1971.

[83] P. Leboeuf. Normal and anomalous diffusion in a deterministic area-preserving map. *Physica D*, 116:8–20, 1998.

[84] F. Ledrappier. Quelques propriétés des exposants caractéristiques. In *École d'été de probabilités de Saint-Flour, XII-1982*, volume 1097 of *Lecture Notes in Math.*, pages 305–396. Springer, Berlin-New York, 1984.

[85] F. Ledrappier. Positivity of the exponents for stationary sequences of matrices. In *Lyapunov exponents (Bremen, 1984)*, volume 1186 of *Lecture Notes in Math.*, pages 56–73. Springer Verlag, Berlin, 1986.

[86] A.J. Lichtenberg and M.A. Lieberman. *Regular and Chaotic Dynamics*, volume 38 of *Applied Mathematical Sciences*. Springer Verlag, New York, second edition, 1992.

[87] D.A. Lind and J.-P. Thouvenot. Measure-preserving homeomorphisms of the torus represent all finite entropy ergodic transformation. *Math. Systems Theory*, 11:275–282, 1977.

[88] R.S. MacKay. Transition to chaos for area-preserving maps. In *Nonlinear dynamics aspects of particle accelerators (Santa Margherita di Pula, 1985)*, volume 247 of *Lecture Notes in Phys.*, pages 390–454. Springer, Berlin-New York, 1986.

[89] R.S. MacKay. *Renormalisation in area-preserving maps*, volume 6 of *Advanced Series in Nonlinear Dynamics*. World Scientific Publishing Co., Inc., River Edge, NJ, 1993.

[90] R.S. MacKay. Recent progress and outstanding problems in Hamiltonian dynamics. *Physica D*, 86:122–133, 1995.

[91] R.S. MacKay and J.D. Meiss. Survey of Hamiltonian dynamics. In *Hamiltonian Dynamical Systems, a reprint selection*. Adam Hilger, Bristol and Philadelphia, 1987.

[92] R. Mané. A proof of Pesin's formula. *Ergod. Th. Dyn. Sys.*, 1:95–102, 1981.

[93] R. Mané. Lyapounov exponents and stable manifolds for compact transformations. In *Geometric dynamics (Rio de Janeiro, 1981)*, pages 522–577. Springer, Berlin-New York, 1983.

[94] R. Mané. *Ergodic theory and differentiable dynamics*. Ergebnisse der Mathematik und ihrer Grenzgebiete; ser.3, Bd. 8. Springer-Verlag, Berlin, 1987.

[95] R. Mané. The Lyapunov exponents of generic area preserving diffeomorphisms. In *International Conference on Dynamical Systems (Montevideo, 1995)*, volume 362 of *Pitman Res. Notes Math. Ser.*, pages 110–119. Longman, Harlow, 1996.

[96] A. Manning. There are no new Anosov diffeomorphisms on tori. *Amer. J. Math.*, 96:422–429, 1974.

[97] J. Mather. Characterization of Anosov diffeomorphisms. *Indag. Math., N.S.*, 30:479–483, 1968.

[98] J. Mather and G. Forni. Action minimizing orbits in Hamiltonian systems. In *Transition to chaos in classical and quantum mechanics (Montecatini Terme, 1991)*, volume 1589 of *Lecture Notes in Math.*, pages 92–186. Springer, Berlin, 1994.

[99] J.N. Mather. Nonexistence of invariant circles. *Ergod. Th. Dyn. Sys.*, 4:301–309, 1984.

[100] J.N. Mather. Dynamics of area preserving maps. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986)*, pages 1190–1194. Amer. Math. Soc., Providence, RI, 1987.

[101] J.D. Meiss. Transient measures in the Standard map. *Phys. D*, 74:254–267, 1994.

[102] M. Misiurewicz and F. Przytycki. Entropy conjecture for tori. *Bull. Acad Polon. Sci. Sér. Sci Math. Astronom. Phys*, 25:575–578, 1977.

[103] L. Mora. Birkhoff-Henon attractors for dissipative perturbations of area-preserving twist maps. *Ergod. Th. Dyn. Sys.*, 14:809–815, 1994.

[104] L. Mora and N. Romero. Moser's invariant curves and homoclinic bifurcations. *Dynam. Systems Appl.*, 6:29–41, 1997.

[105] J. Moser. *Stable and random Motion in dynamical systems*. Princeton University Press, Princeton, 1973.

[106] J. Moser. Monotone twist mappings and the calculus of variations. *Ergod. Th. Dyn. Sys.*, 6:401–413, 1986.

[107] J.F. Murray and J. von Neumann. On rings of operators. IV. *Annals of Mathematics*, 44:716–808, 1943.

[108] A.I. Neishtadt, V.V. Sidorenko, and D.V. Treschev. Stable periodic motions in the problem on passage through a separatrix. *Chaos*, 7:2–11, 1997.

[109] S.E. Newhouse. Quasi-elliptic periodic points in conservative dynamical systems. *Amer. J. Math.*, 99:1061–1087, 1975.

[110] S.E. Newhouse. Continuity properties of entropy. *Annals of Mathematics*, 129:215–235, 1989. also Ann. of Math. 131 (1990) 409-410.

[111] J.C. Oxtoby and S.M. Ulam. Measure-preserving homeomorphisms and metrical transitivity. *Annals of Mathematics*, 42:874–920, 1941.

[112] J. Palis. Homoclinic bifurcations, sensitive-chaotic dynamics and strange attractors. In *Dynamical systems and related topics (Nagoya, 1990)*, pages 466–472. World Sci. Publishing, River Edge, NJ, 1991.

[113] J. Palis. Seminar talk given at ETH, 1993.

[114] J. Palis and F. Takens. *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations*, volume 35 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1993.

[115] M. Paramio and J. Sesma. A continued fraction procedure to determine invariant directions in twist mappings. *Phys. Lett. A*, 117:333–336, 1986.

[116] L. Pastur and A.Figotin. *Spectra of Random and Almost-Periodic Operators*, volume 297. Springer-Verlag, Berlin–New York, Grundlehren der Mathematischen Wissenschaften, 1992.

[117] Ya.B. Pesin. Characteristic Lyapunov exponents and smooth ergodic theory. *Russ. Math. Surveys*, 32:55–114, 1977.

[118] M. Pollicott. *Lectures on ergodic theory and Pesin theory on compact manifolds*, volume 180 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1993.

[119] F. Przytycki. Examples of conservative diffeomorphisms of the two dimensional torus with coexistence of elliptic and stochastic behaviour. *Ergod. Th. Dyn. Sys.*, 2:439–463, 1982.

[120] F. Rannou. Numerical study of discrete area-preserving mappings. *Astron. Astroph.*, 31:289–301, 1974.

[121] T. Ransford. *Potential theory in the complex plane*, volume 28 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1995.

[122] A.B. Rechester and R.B. White. Calculation of turbulent diffusion for the Chirikov-Taylor model. *Phys. Rev. Lett.*, 44:1586–1589, 1980.

[123] R.Johnson and J.Moser. The rotation number for almost periodic potentials. *Commun. Math. Phys.*, 84:403–438, 1982.

[124] D. Ruelle. Analyticity properties of the characteristic exponents of random matrix products. *Adv. in Math.*, 32:68–80, 1979.

[125] D. Ruelle. Ergodic theory of differentiable dynamical systems. *Pub. I.H.E.S.*, 50:27–58, 1979.

[126] D. Ruelle. Characteristic exponents and invariant manifolds in Hilbert space. *Annals of Mathematics*, 115:243–290, 1982.

[127] D. Ruelle. Rotation numbers for diffeomorphisms and flows. *Ann. Inst. Henri Poincaré Phys. Théor.*, 42:109–115, 1985.

[128] D. Ruelle. Differentiation of SRB states. *Commun. Math. Phys.*, 187:227–241, 1997.

[129] B. Simon. *Trace Ideals and their applications*. London Mathematical Society Lecture Notes Series, 35. Cambridge University Press, Cambridge, 1979.

[130] B. Simon. Operators with singular continuous spectrum: I. General operators. *Annals of Mathematics*, 141:131–145, 1995.

[131] Ya. G. Sinai. *Topics in ergodic theory*, volume 44 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1994.

[132] Ya. G. Sinai. Talk at the UCLA conference 'Quo Vadis', 1996.

[133] E. Sorets and T. Spencer. Positive Lyapunov exponents for Schrödinger operators with quasi-periodic potentials. *Commun. Math. Phys.*, 142:543–566, 1991.

[134] T. Spencer. Some rigorous results for random and quasiperiodic potentials. *Physica A*, 140:70–77, 1989.

[135] A. Stirnemann. Renormalization for golden circles. *Commun. Math. Phys.*, 152:369–431, 1993.

[136] A. Stirnemann. Towards an existence proof of MacKay's fixed point. *Commun. Math. Phys.*, 188:723–735, 1997.

[137] J.-M. Strelcyn. The coexistence problem for conservative dynamical systems: A review. *Colloq. Math.*, 62:331–345, 1991.

[138] S. Surace. Positive Lyapunov exponents for a class of ergodic Schrödinger operators. *Commun. Math. Phys.*, 162:529–537, 1994.

[139] Y. Suris. Integrable mappings of the Standard type. *Funct. Anal. Appl.*, 23:74–76, 1989.

[140] F. Takens. Homoclinic points in conservative systems. *Inv. Math.*, 18:267–292, 1972.

[141] D. Umberger and D. Farmer. Fat fractals on the energy surface. *Phys. Rev. Lett.*, 55:661–664, 1985.

[142] M. Viana. Dynamics: a probabilistic and geometric perspective. *Doc. Math.J. DMV*, I:557–578, 1998. Extra Volume ICM, 1998.

[143] P. Walters. *An introduction to ergodic theory*. Graduate texts in mathematics 79. Springer-Verlag, New York, 1982.

[144] E. Wayne. Coherent behavior in large systems of coupled chaotic oscillator. Talk given at UT, March, 1999.

[145] A.S. Wightman. The mechanisms of stochasticity in classical dynamical systems. In *Perspectives in statistical physics*, Stud. Statist. Mech., IX, pages 343–363. North-Holland, Amsterdam-New York, 1981.

[146] M. Wojtkowski. A model problem with the coexistence of stochastic and integrable behaviour. *Commun. Math. Phys.*, 80:453–464, 1981.

[147] M. Wojtkowski. On the ergodic properties of piecewise linear perturbations of the twist map. *Ergod. Th. Dyn. Sys.*, 2:525–542, 1982.

[148] M. Wojtkowski. Invariant families of cones and Lyapunov exponents. *Ergod. Th. Dyn. Sys.*, 5:145–161, 1985.

[149] M. Wojtkowski. Principles for the design of billiards with nonvanishing Lyapunov exponents. *Commun. Math. Phys.*, 105:391–414, 1986.

[150] S. Wolfram. *Mathematica-A system for doing mathematics by computer.* Addison-Wesley, 1991.

[151] Z. J. Xia. Topological entropy and stability. Tagungsbericht, 27/1997, Dynamische Systeme, 13.7. bis 19.7. 1997, Oberwohlfach, 1997.

[152] J-C. Yoccoz. Introduction to hyperbolic dynamics. In *Real and complex dynamical systems (Hillerød, 1993)*, volume 464 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 265–291. Kluwer Acad. Publ., Dordrecht, 1995.

[153] L.-S. Young. Dimension, entropy and Lyapunov exponents. *Ergod. Th. Dyn. Sys.*, 2:109–124, 1982.

[154] L.-S. Young. Some open sets of nonuniformly hyperbolic cocycles. *Ergod. Th. Dyn. Sys.*, 13:409–415, 1993.

[155] L.-S. Young. Ergodic theory of differentiable dynamical systems. In *Real and complex dynamical systems (Hillerød, 1993)*, volume 464 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 293–336. Kluwer Acad. Publ., Dordrecht, 1995.

[156] L.-S. Young. Lyapunov exponents for some quasi-periodic cocycles. *Ergod. Th. Dyn. Sys.*, 17:483–504, 1997.

[157] E. Zehnder. Homoclinic points near elliptic fixed points. *Commun. Pure Appl. Math.*, 26:131–182, 1973.

[158] X-S. Zhang and F. Vivaldi. Small perturbations of a discrete twist map. *Ann. Inst. H. Poincaré Phys. Théor.*, 68:507–523, 1998.